Lecture (10): Partial differential equations

Derivation of the diffusion equation

To derive the diffusion equation in one spacial dimension, we imagine a still liquid in a long pipe of constant cross sectional area. A small quantity of dye is placed in a cross section of the pipe and allowed to diffuse up and down the pipe. The dye diffuses from regions of higher concentration to regions of lower concentration.

We define $u(x,t)$ to be the concentration of the dye at position x along the pipe, and we wish to find the pde satisfied by u. It is useful to keep track of the units of the various quantities involved in the derivation and we introduce the bracket notation $[X]$ to mean the units of X. Relevant dimensional units used in the derivation of the diffusion equation are mass m, length l, and time t. Assuming that the dye concentration is uniform in every cross section of the pipe, the dimensions of concentration used here are $[u] = m/l$.

The mass of dye in the infinitesimal pipe volume located between position x_1 and position x_2 at time t, with $x_1 < x < x_2$, is given to order $\Delta x = x_2 - x_1$ by

$$M = u(x,t)\Delta x.$$

The mass of dye in this infinitesimal pipe volume changes by diffusion into and out of the cross sectional ends situated at position x_1 and x_2 (Figure 8.1). We assume the rate of diffusion is proportional to the concentration gradient, a relationship known as Fick's law of diffusion. Fick's law of diffusion assumes the mass flux J, with units $[J] = m/t$ across a cross section of the pipe is given

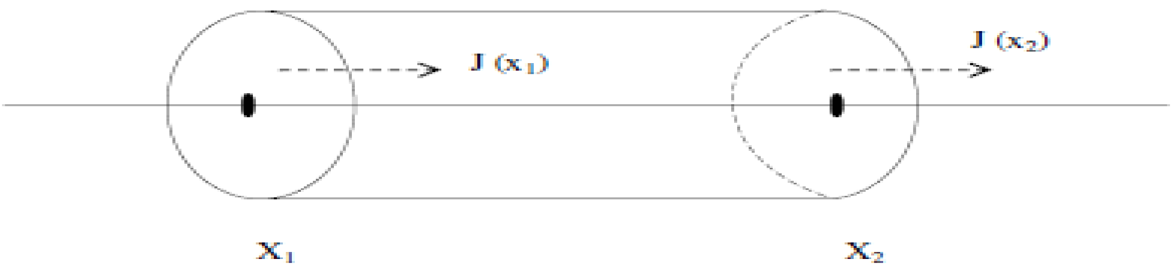

Figure 8.1: *Derivation of the diffusion equation.*

by

$$J = -Du_x, \tag{8.1}$$

where the diffusion constant $D > 0$ has units $[D] = l^2/t$, and we have used the notation $u_x = \partial u/\partial x$. The mass flux is opposite in sign to the gradient of concentration. The time rate of change in the mass of dye between x_1 and x_2 is given by the difference between the mass flux into and the mass flux out of the infinitesimal cross sectional volume. When $u_x < 0$, $J > 0$ and the mass flows into the volume at position x_1 and out of the volume at position x_2. On the other hand, when $u_x > 0$, $J < 0$ and the mass flows out of the volume at position x_1 and into the volume at position x_2. In both cases, the time rate of change of the dye mass is given by

$$\frac{dM}{dt} = J(x_1,t) - J(x_2,t),$$

or rewriting in terms of $u(x,t)$:

$$u_t(x,t)\Delta x = D\left(u_x(x_2,t) - u_x(x_1,t)\right).$$

Dividing by Δx and taking the limit $\Delta x \to 0$ results in the diffusion equation:

$$u_t = Du_{xx}.$$

We note that the diffusion equation is identical to the heat conduction equation, where u is temperature, and the constant D (commonly written as κ) is the thermal conductivity.

2- Derivation of the wave equation

To derive the wave equation in one spacial dimension, we imagine an elastic string that undergoes small amplitude transverse vibrations. We define $u(x, t)$ to be the vertical displacement of the string from the x-axis at position x and time t, and we wish to find the pde satisfied by u. We define ρ to be the constant mass density of the string, T the tension of the string, and θ the angle

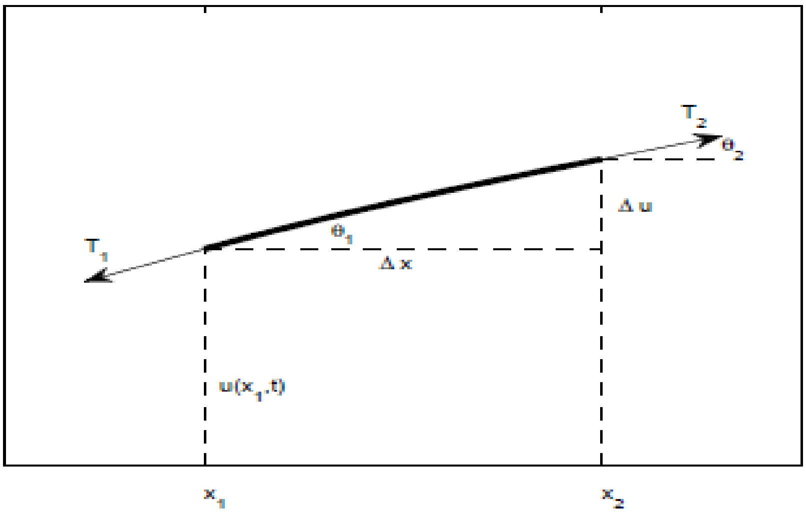

Figure 8.2: *Derivation of the wave equation.*

between the string and the horizonal line. We consider an infinitesimal string element located between x_1 and x_2, with $\Delta x = x_2 - x_1$, as shown in Fig. 8.2. The governing equations are Newton's law of motion for the horizontal and vertical acceleration of our infinitesimal string element, and we assume that the string element only accelerates vertically. Therefore, the horizontal forces must balance and we have

$$T_2 \cos\theta_2 = T_1 \cos\theta_1.$$

The vertical forces result in a vertical acceleration, and with u_{tt} the vertical acceleration of the string element and $\rho\sqrt{\Delta x^2 + \Delta u^2} = \rho\Delta x\sqrt{1+u_x^2}$ its mass, where we have used $u_x = \Delta u/\Delta x$, exact as $\Delta x \to 0$, we have

$$\rho\Delta x\sqrt{1+u_x^2}\,u_{tt} = T_2 \sin\theta_2 - T_1 \sin\theta_1.$$

We now make the assumption of small vibrations, that is $\Delta u \ll \Delta x$, or equivalently $u_x \ll 1$. Note that $[u] = l$ so that u_x is dimensionless. With this approximation, to leading-order in u_x we have

$$\cos\theta_2 = \cos\theta_1 = 1,$$

$$\sin\theta_2 = u_x(x_2, t), \quad \sin\theta_1 = u_x(x_1, t),$$

and

$$\sqrt{1+u_x^2} = 1.$$

Therefore, to leading order $T_1 = T_2 = T$, (i.e., the tension in the string is approximately constant), and

$$\rho\Delta x u_{tt} = T\left(u_x(x_2, t) - u_x(x_1, t)\right).$$

Dividing by Δx and taking the limit $\Delta x \to 0$ results in the wave equation

$$u_{tt} = c^2 u_{xx},$$

where $c^2 = T/\rho$. Since $[T] = ml/t^2$ and $[\rho] = m/l$, we have $[c^2] = l^2/t^2$ so that c has units of velocity.