Lecture (12) : Fourier series

Our solution of the diffusion and wave equations will require use of a Fourier series. A periodic function $f(x)$ with period $2L$, can be represented as a Fourier series in the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right). \tag{8.2}$$

Determination of the coefficients $a_0, a_1, a_2, \ldots$ and $b_1, b_2, b_3, \ldots$ makes use of orthogonality relations for sine and cosine. We first define the widely used Kronecker delta δ_{nm} as

$$\delta_{nm} = \begin{cases} 1 & \text{if } n = m; \\ 0 & \text{otherwise.} \end{cases}$$

The orthogonality relations for n and m positive integers are then given with compact notation as the integration formulas

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \cos\left(\frac{n\pi x}{L}\right) dx = L\delta_{nm}, \tag{8.3}$$

$$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = L\delta_{nm}, \tag{8.4}$$

$$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = 0. \tag{8.5}$$

We illustrate the integration technique used to obtain these results. To derive (8.4), we assume that n and m are positive integers with $n \neq m$, and we make use of the change of variables $\xi = \pi x/L$:

$$\begin{aligned}
&\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx \\
&\quad = \frac{L}{\pi} \int_{-\pi}^{\pi} \sin(m\xi) \sin(n\xi) d\xi \\
&\quad = \frac{L}{2\pi} \int_{-\pi}^{\pi} \left[\cos((m-n)\xi) - \cos((m+n)\xi)\right] d\xi \\
&\quad = \frac{L}{2\pi} \left[\frac{1}{m-n} \sin((m-n)\xi) - \frac{1}{m+n} \sin((m+n)\xi)\right]_{-\pi}^{\pi} \\
&\quad = 0.
\end{aligned}$$

For $m = n$, however,

$$\begin{aligned}
\int_{-L}^{L} \sin^2\left(\frac{n\pi x}{L}\right) dx &= \frac{L}{\pi} \int_{-\pi}^{\pi} \sin^2(n\xi) d\xi \\
&= \frac{L}{2\pi} \int_{-\pi}^{\pi} \left(1 - \cos(2n\xi)\right) d\xi \\
&= \frac{L}{2\pi} \left[\xi - \frac{1}{2n} \sin 2n\xi\right]_{-\pi}^{\pi} \\
&= L.
\end{aligned}$$

Integration formulas given by (8.3) and (8.5) can be similarly derived.
To determine the coefficient an, we multiply both sides of (8.2) by cos $(n\pi x/L)$ with n a nonnegative integer, and change the dummy summation variable from n to m. Integrating over x from $-L$ to L and assuming that the integration can be done term by term in the infinite sum, we obtain

$$\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = \frac{a_0}{2}\int_{-L}^{L}\cos\frac{n\pi x}{L}dx + \sum_{m=1}^{\infty}\left\{a_m\int_{-L}^{L}\cos\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx + b_m\int_{-L}^{L}\cos\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx\right\}.$$

If □ = 0, then the second and third integrals on the right-hand-side are zero and the first integral is 2 L so that the right-hand-side becomes La_0.. If n is a positive integer, then the first and third integrals on the right-hand-side are zero, and the second integral is $L\delta_{nm}$. For this case, we have

$$\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx = \sum_{m=1}^{\infty} La_m\delta_{nm} = La_n,$$

where all the terms in the summation except $m = n$ are zero by virtue of the Kronecker delta. We therefore obtain for $n = 0, 1, 2, \ldots$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx.$$

To determine the coefficients $b1$, $b2$, $b3$, . . . , we multiply both sides of (8.2) by sin $(n\pi x/L)$, with n a positive integer, and again change the dummy summation variable from n to m. Integrating, we obtain

$$\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = \frac{a_0}{2}\int_{-L}^{L}\sin\frac{n\pi x}{L}dx + \sum_{m=1}^{\infty}\left\{a_m\int_{-L}^{L}\sin\frac{n\pi x}{L}\cos\frac{m\pi x}{L}dx + b_m\int_{-L}^{L}\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx\right\}.$$

Here, the first and second integrals on the right-hand-side are zero, and the third integral is $L\delta_{nm}$ so that

$$\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx = \sum_{m=1}^{\infty} Lb_m\delta_{nm} = Lb_n.$$

Hence, for $n = 1, 2, 3, \ldots,$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx. \tag{8.7}$$

Our results for the Fourier series of a function $f(x)$ with period $2L$ are thus given by (8.2), (8.6) and (8.7).

3- Fourier cosine and sine series

The Fourier series simplifies if $f(x)$ is an even function such that $f(-x) = f(x)$, or an odd function such that $f(-x) = -f(x)$. Use will be made of the following facts. The function cos $(n\pi x/L)$ is an even function and sin $(n\pi x/L)$ is an odd function. The product of two even functions is an even function. The product of two odd functions is an even function. The product of an even and an odd function is an odd function. And if $g(x)$ is an even function, then

$$\int_{-L}^{L} g(x)dx = 2\int_{0}^{L} g(x)dx;$$

and if $g(x)$ is an odd function, then

$$\int_{-L}^{L} g(x)dx = 0.$$

We examine in turn the Fourier series for an even or an odd function. First, if $f(x)$ is even, then from (8.6) and (8.7) and our facts about even and odd functions,

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}dx, \quad b_n = 0. \tag{8.8}$$

The Fourier series for an even function with period $2L$ is thus given by the Fourier cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \quad f(x) \text{ even.} \tag{8.9}$$

Second, if $f(x)$ is odd, then

$$a_n = 0, \quad b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx; \tag{8.10}$$

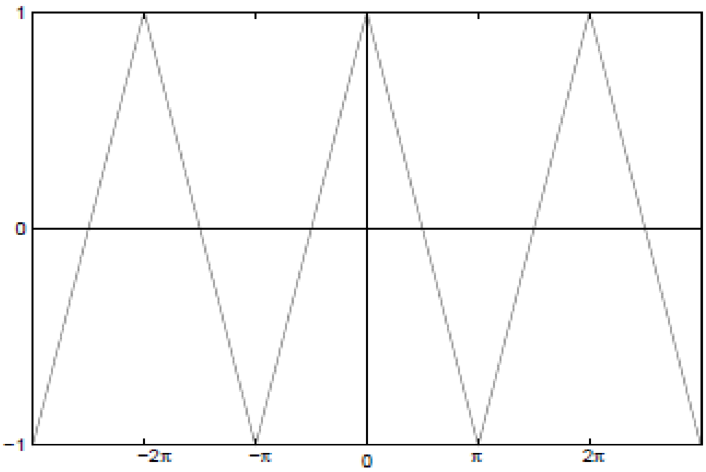

Figure 8.3: *The even triangle function.*

and the Fourier series for an odd function with period $2L$ is given by the Fourier sine series

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad f(x) \text{ odd}. \tag{8.11}$$

Examples of Fourier series computed numerically can be obtained using the Java applet found at http://www.falstad.com/fourier. Here, we demonstrate an analytical example.