Lecture (13):

Example: Determine the Fourier cosine series of the even triangle function represented by Fig. 8.3.

The triangle function depicted in Fig. 8.3 is an even function of x with period 2π (i.e., $L = \pi$). Its definition on $0 < x < \pi$ is given by

$$f(x) = 1 - \frac{2x}{\pi}.$$

Because $f(x)$ is even, it can be represented by the Fourier cosine series given by (8.8) and (8.9). The coefficient a_0 is

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^\pi f(x)dx \\ &= \frac{2}{\pi}\int_0^\pi \left(1 - \frac{2x}{\pi}\right) dx \\ &= \frac{2}{\pi}\left[x - \frac{x^2}{\pi}\right]_0^\pi \\ &= 0. \end{aligned}$$

The coefficients for $n > 0$ are

$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)dx \\ &= \frac{2}{\pi}\int_0^\pi \left(1 - \frac{2x}{\pi}\right)\cos(nx)dx \\ &= \frac{2}{\pi}\int_0^\pi \cos(nx)dx - \frac{4}{\pi^2}\int_0^\pi x\cos(nx)dx \\ &= \frac{2}{n\pi}\sin(nx)\Big]_0^\pi - \frac{4}{\pi^2}\left\{\left[\frac{x}{n}\sin(nx)\right]_0^\pi - \frac{1}{n}\int_0^\pi \sin(nx)dx\right\} \\ &= \frac{4}{n\pi^2}\int_0^\pi \sin(nx)dx \\ &= -\frac{4}{n^2\pi^2}\cos(nx)\Big]_0^\pi \\ &= \frac{4}{n^2\pi^2}(1 - \cos(n\pi)). \end{aligned}$$

Since

$$\cos(n\pi) = \begin{cases} -1, & \text{if } n \text{ odd;} \\ 1, & \text{if } n \text{ even;} \end{cases}$$

we have

$$a_n = \begin{cases} 8/(n^2\pi^2), & \text{if } n \text{ odd;} \\ 0, & \text{if } n \text{ even.} \end{cases}$$

The Fourier cosine series for the triangle function is therefore given by

$$f(x) = \frac{8}{\pi^2}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right).$$

Convergence of this series is rapid. As an interesting aside, evaluation of this series at $x = 0$, using $f(0) = 1$, yields an infinite series for $\pi^2/8$:

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots .$$

With Fourier series now included in our applied mathematics toolbox, we are ready to solve the diffusion and wave equations in bounded domains.