LECTUER (3):

Exact Equations

Defining Properties

$$M(x, y)dx + N(x, y)dy = 0 \ \ldots\ldots. \ (2.11)$$

$$dg(x, y) = M(x, y)dx + N(x, y)dy \ldots\ldots.. \ (2.12)$$

The differential equation *M(x,y)dx* + *N(x,y)dy* = 0 is an exact equation if : $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

where : $\partial F/\partial x = M(x,y)$ and $\partial F/\partial y = N(x,y)$

Solution :

1. Integrate either *M(x,y)* with respect to *x* or *N(x,y)* to *y*. Assume integrating *M(x,y),* then :

$$F(x,y) = \int M(x,y)dx + \theta(y)$$

2. Now : $$\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\left[\int M(x,y)dx\right] + \theta'(y) = N(x,y)$$

or : $$\theta'(y) = N(x,y) - \frac{\partial}{\partial y}\left[\int M(x,y)dx\right]$$

3. Integrate *θ'(y)* to get *θ(y)* and write down the result *F(x,y)* = *C*

Note!

Test for exactness: If $M(x,y)$ and $N(x,y)$ are continuous functions and have continuous first partial derivatives on some rectangle of the xy-plane, then Equation 2.11 is exact if and only if

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x} \qquad (2.13)$$

Method of Solution

To solve Equation 2.11, assuming that it is exact, first solve the equations

$$\frac{\partial g(x,y)}{\partial x} = M(x,y) \qquad (2.14)$$

$$\frac{\partial g(x,y)}{\partial y} = N(x,y) \qquad (2.15)$$

for $g(x, y)$. The solution to 2.11 is then given implicitly by

$$g(x, y) = c \tag{2.16}$$

where c represents an arbitrary constant.

Equation 2.16 is immediate from Equations 2.11 and 2.12. If 2.12 is substituted into 2.11, we obtain $dg(x, y(x)) = 0$. Integrating this equation (note that we can write 0 as $0\,dx$), we have $\int dg(x, y(x)) = \int 0dx$,

Example1 : Determine whether the differential equation

$ydx - xdy = 0$ is exact.

This equation has the form of Equation 2.11 with $M(x, y) = y$ and $N(x, y) = -x$. Here

$$\frac{\partial M}{\partial y} = 1 \quad \text{and} \quad \frac{\partial N}{\partial x} = -1$$

which are not equal, so the differential equation is not exact.

Example2: Determine whether $-1/x^2$ is an integrating factor for the differential equation $ydx - xdy = 0$.

It was shown in Example1that the differential equation is not exact. Multiplying it by $-1/x^2$, we obtain

$$\frac{-1}{x^2}(ydx - xdy) = 0 \quad \text{or} \quad \frac{-y}{x^2}dx + \frac{1}{x}dy = 0 \qquad (2.28)$$

Equation 2.28 has the form of Equation 2.11 with $M(x, y) = -y/x^2$ and $N(x, y) = 1/x$. Now

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2}\right) = \frac{-1}{x^2} = \frac{\partial}{\partial x}\left(\frac{1}{x}\right) = \frac{\partial N}{\partial x}$$

so 2.28 is exact, which implies that $-1/x^2$ is an integrating factor for the original differential equation.

Solved Problem 2.6 Solve $ydx - xdy = 0$.

Using the results of Problem 2.5, we can rewrite the given differential equation as

$$\frac{xdy - ydx}{x^2} = 0$$

which is exact. Equation 2.28 can be solved using the steps described in Equations 2.14 through 2.16.

Alternatively, we note from Table 2.1 that 2.28 can be rewritten as $d(y/x) = 0$. Hence, by direct integration, we have $y/x = c$, or $y = xc$, as the solution.