LECTUER (4) : NOT EXACT (Integrating Factors)

In general, Equation 2.11 is not exact. Occasionally, it is possible to transform 2.11 into an exact differential equation by a judicious multiplication. A function $I(x, y)$ is an *integrating factor* for 2.11 if the equation

$$I(x, y)[M(x, y)dx + N(x, y)dy] = 0 \qquad (2.17)$$

is exact. A solution to 2.11 is obtained by solving the exact differential equation defined by 2.17. Some of the more common integrating factors are displayed in Table 2.1 and the conditions that follow:

If $\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \equiv g(x)$,

a function of x alone, then

$$I(x, y) = e^{\int g(x)dx} \qquad (2.18)$$

If $\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) \equiv h(y)$

a function of y alone, then

$$I(x, y) = e^{-\int h(y)dy} \qquad (2.19)$$

If $M = yf(xy)$ and $N = xg(xy)$, then

$$I(x, y) = \frac{1}{xM - yN} \qquad (2.20)$$

In general, integrating factors are difficult to uncover. If a differential equation does not have one of the forms given above, then a search for an integrating factor likely will not be successful, and other methods of solution are recommended.

(See Problems 2.3–2.6)

Linear Equations

Method of Solution

A first-order *linear* differential equation has the form (see Chapter One)

$$y' + p(x)y = q(x) \tag{2.21}$$

An integrating factor for Equation 2.21 is

$$I(x) = e^{\int p(x)dx} \tag{2.22}$$

Table 1.1

Group of terms	Integrating factor $I(x, y)$	Exact differential $dy(x, y)$
$y\,dx - x\,dy$	$-\dfrac{1}{x^2}$	$\dfrac{x\,dy - y\,dx}{x^2} = d\left(\dfrac{y}{x}\right)$
$y\,dx - x\,dy$	$\dfrac{1}{y^2}$	$\dfrac{y\,dx - x\,dy}{y^2} = d\left(\dfrac{x}{y}\right)$
$y\,dx - x\,dy$	$-\dfrac{1}{xy}$	$\dfrac{x\,dy - y\,dx}{xy} = d\left(\ln \dfrac{y}{x}\right)$
$y\,dx - x\,dy$	$-\dfrac{1}{x^2 + y^2}$	$\dfrac{x\,dy - y\,dx}{x^2 + y^2} = d\left(\arctan \dfrac{y}{x}\right)$
$y\,dx + x\,dy$	$\dfrac{1}{xy}$	$\dfrac{y\,dx + x\,dy}{xy} = d(\ln xy)$
$y\,dx + x\,dy$	$\dfrac{1}{(xy)^n}, \quad n > 1$	$\dfrac{y\,dx + x\,dy}{(xy)^n} = d\left[\dfrac{-1}{(n-1)(xy)^{n-1}}\right]$
$y\,dy + x\,dx$	$\dfrac{1}{x^2 + y^2}$	$\dfrac{y\,dy + x\,dx}{x^2 + y^2} = d\left[\dfrac{1}{2}\ln (x^2 + y^2)\right]$
$y\,dy + x\,dx$	$\dfrac{1}{(x^2 + y^2)^n}, \quad n > 1$	$\dfrac{y\,dy + x\,dx}{(x^2 + y^2)^n} = d\left[\dfrac{-1}{2(n-1)(x^2 + y^2)^{n-1}}\right]$
$ay\,dx + bx\,dy$ (a, b constants)	$x^{a-1}y^{b-1}$	$x^{a-1}y^{b-1}(ay\,dx + bx\,dy) = d(x^a y^b)$

which depends only on x and is independent of y. When both sides of 2.21 are multiplied by $I(x)$, the resulting equation

$$I(x)y' + p(x)I(x)y = I(x)q(x) \tag{2.23}$$

is exact. This equation can be solved by the method described previously. A simpler procedure is to rewrite 2.23 as

$$\frac{d(yI)}{dx} = Iq(x)$$

integrate both sides of this last equation with respect to x, and then solve the resulting equation for y. The general solution for Equation 2.21 is

$$y = \frac{\int I(x)q(x)dx + c}{I(x)}$$

where c is the constant of integration.

Bernoulli Equations

A Bernoulli differential equation has the form

$$y' + p(x)y = q(x)y^n \tag{2.24}$$

where n is a real number. The substitution

$$z = y^{1-n} \quad (2.25$$