LECTUER (5): Solved Problems

Example: Solve $2xydx + (1 + x^2)dy = 0$.

This equation has the form of Equation 2.11 with $M(x, y) = 2xy$ and $N(x, y) = 1 + x^2$. Since $\partial M / \partial y = \partial N / \partial x = 2x$, the differential equation is exact. Because this equation is exact, we now determine a function $g(x, y)$ that satisfies Equations 2.14 and 2.15. Substituting $M(x, y) = 2xy$ into 2.14, we obtain $\partial g / \partial x = 2xy$. Integrating both sides of this equation with respect to x, we find

$$\int \frac{\partial g}{\partial x} dx = \int 2xydx$$

or

$$g(x, y) = x^2 y + h(y) \qquad (2.27)$$

Note that when integrating with respect to x, the constant (*with respect to* x) of integration can depend on y.

We now determine $h(y)$. Differentiating 2.27 with respect to y, we obtain $\partial g / \partial y = x^2 + h'(y)$ Substituting this equation along with $N(x, y) = 1 + x^2$ into 2.15, we have

$$x^2 + h'(y) = 1 + x^2 \text{ or } h'(y) = 1$$

Integrating this last equation with respect to y, we obtain $h(y) = y + c_1$ (c_1 = constant). Substituting this expression into 2.27 yields

$$g(x, y) = x^2 y + y + c_1$$

The solution to the differential equation, which is given implicitly by 2.16 as $g(x, y) = c$, is

$$x^2 y + y = c_2 \qquad (c_2 = c - c_1)$$

Solving for y explicitly, we obtain the solution as $y = c_2/(x^2 + 1)$.

Example: Find G.S for D. E. $y' + (4 / x)y = x^4$.

The differential equation has the form of Equation 2.21, with $p(x) = 4/ x$ and $q(x) = x^4$, and is linear. Here

$$\int p(x)dx = \int \frac{4}{x} dx = 4 \ln |x| = \ln x^4$$

so 2.22 becomes

$$I(x) = e^{\int p(x)dx} = e^{\ln x^4} = x^4 \tag{2.29}$$

Multiplying the differential equation by the integrating factor defined by 2.29, we obtain

$$x^4 y' + 4x^3 y = x^8 \text{ or } \frac{d}{dx}(yx^4) = x^8$$

Integrating both sides of this last equation with respect to x, we obtain

$$yx^4 = \frac{1}{9}x^9 + c \text{ or } y = \frac{c}{x^4} + \frac{1}{9}x^5$$

Example : Solve $y' + xy = xy^2$.

This equation is not linear. It is, however, a Bernoulli differential equation having the form of Equation 2.24 with $p(x) = q(x) = x$, and $n = 2$. We make the substitution suggested by 2.25, namely $z = y^{1-2} = y^{-1}$, from which follow

$$y = \frac{1}{z} \text{ and } y' = -\frac{z'}{z^2}$$

Substituting these equations into the differential equation, we obtain

$$-\frac{z'}{z^2} + \frac{x}{z} = \frac{x}{z^2} \text{ or } z' - xz = -x$$

This last equation is linear for the unknown function $z(x)$. It has the form of Equation 2.21 with y replaced by z and $p(x) = q(x) = -x$. The integrating factor is

$$I(x) = e^{\int (-x)dx} = e^{-x^2/2}$$

Multiplying the differential equation by $I(x)$, we obtain

$$e^{-x^2/2}\frac{dz}{dx} - xe^{-x^2/2}z = -xe^{-x^2/2}$$

or

$$\frac{d}{dx}\left(ze^{-x^2/2}\right) = -xe^{-x^2/2}$$

Upon integrating both sides of this last equation, we have

$$ze^{-x^2/2} = e^{-x^2/2} + c$$

whereupon

$$z(x) = ce^{x^2/2} + 1$$

The solution of the original differential equation is then

$$y = \frac{1}{z} = \frac{1}{ce^{x^2/2} + 1}$$