LECTUER (6):

Chapter 3

Applications of First-Order Differential Equations

In This Chapter:

- ✔ *Growth and Decay Problems*
- ✔ *Temperature Problems*
- ✔ *Falling Body Problems*
- ✔ *Dilution Problems*
- ✔ *Electrical Circuits*
- ✔ *Orthogonal Trajectories*
- ✔ *Solved Problems*

Growth and Decay Problems

Let $N(t)$ denote the amount of substance (or population) that is either growing or decaying. If we assume that dN/dt, the time rate of change of this amount of substance, is proportional to the amount of substance present, then $dN/dt = kN$, or

$$\frac{dN}{dt} - kN = 0 \tag{3.1}$$

where k is the constant of proportionality.

We are assuming that $N(t)$ is a differentiable, hence continuous, function of time. For population problems, where $N(t)$ is actually discrete and integer-valued, this assumption is incorrect. Nonetheless, 3.1 still provides a good approximation to the physical laws governing such a system.

Temperature Problems

Newton's law of cooling, which is equally applicable to heating, states that *the time rate of change of the temperature of a body is proportional to the temperature difference between the body and its surrounding medium*. Let T denote the temperature of the body and let T_m denote the temperature of the surrounding medium. Then the time rate of change of the temperature of the body is dT/dt, and Newton's law of cooling can be formulated as $dT/dt = -k(T - T_m)$, or as

$$\frac{dT}{dt} + kT = kT_m \tag{3.2}$$

where k is a *positive* constant of proportionality. Once k is chosen positive, the minus sign is required in Newton's law to make dT / dt negative in a cooling process, when T is greater than T_m, and positive in a heating process, when T is less than T_m.

Falling Body Problems

Consider a vertically falling body of mass m that is being influenced only by gravity g and an air resistance that is proportional to the velocity of the body. Assume that both gravity and mass remain constant and, for convenience, choose the downward direction as the positive direction.

You Need to Know ✓

Newton's second law of motion: *The net force acting on a body is equal to the time rate of change of the momentum of the body* ; *or*, *for constant mass*,

$$F = m\frac{dv}{dt} \tag{3.3}$$

where F is the net force on the body and v is the velocity of the body, *both at time t*.

For the problem at hand, there are two forces acting on the body: the force due to gravity given by the weight w of the body, which equals mg, and the force due to air resistance given by $-kv$, where $k \geq 0$ is a constant of proportionality. The minus sign is required because this force opposes the velocity; that is, it acts in the upward, or negative, direction (see Figure 3-1). The net force F on the body is, therefore, $F = mg - kv$. Substituting this result into 3.3, we obtain

$$mg - kv = m\frac{dv}{dt}$$

or

$$\frac{dv}{dt} + \frac{k}{m}v = g \tag{3.4}$$

as the equation of motion for the body.

If air resistance is negligible or nonexistent, then $k = 0$ and 3.4 simplifies to

$$\frac{dv}{dt} = g \tag{3.5}$$

When $k > 0$, the limiting velocity v_l is defined by

$$\gamma = \frac{mg}{k} \tag{3.6}$$

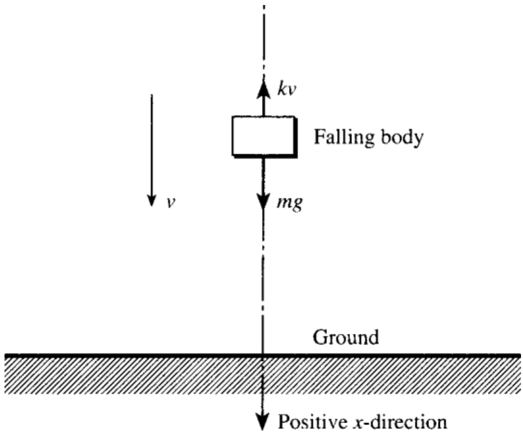

Figure 3-1

Caution: Equations 3.4, 3.5, and 3.6 are valid only if the given conditions are satisfied. These equations are not valid if, for example, air resistance is not proportional to velocity but to the velocity squared, or if the upward direction is taken to be the positive direction.

Dilution Problems

Consider a tank which initially holds V_0 gal of brine that contains a lb of salt. Another solution, containing b lb of salt per gallon, is poured into the tank at the rate of e gal/min while simultaneously, the well-stirred solution leaves the tank at the rate of f gal/min (Figure 3-2). The problem is to find the amount of salt in the tank at any time t.

Let Q denote the amount (in pounds) of salt in the tank at any time. The time rate of change of Q, dQ / dt, equals the rate at which salt enters the tank minus the rate at which salt leaves the tank. Salt enters the tank at the rate of be lb/min. To determine the rate at which salt leaves the tank, we first calculate the volume of brine in the tank at any time t, which is the initial volume V_0 plus the volume of brine added et minus the volume of brine removed ft. Thus, the volume of brine at any time is

$$V_0 + et - ft \tag{3.7}$$

The concentration of salt in the tank at any time is $Q/(V_0 + et - ft)$, from which it follows that salt leaves the tank at the rate of

$$f\left(\frac{Q}{V_0 + et - ft}\right) \text{ lb/min}$$

Thus,

$$\frac{dQ}{dt} = be - f\left(\frac{Q}{V_0 + et - ft}\right)$$

or

$$\frac{dQ}{dt} + \frac{f}{V_0 + (e - f)t}Q = be \tag{3.8}$$

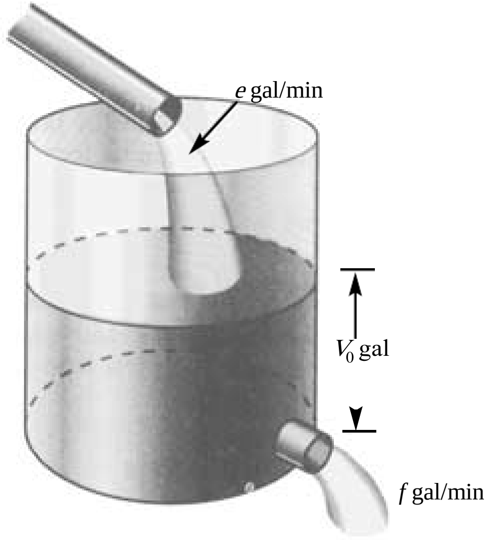

Figure 3-2

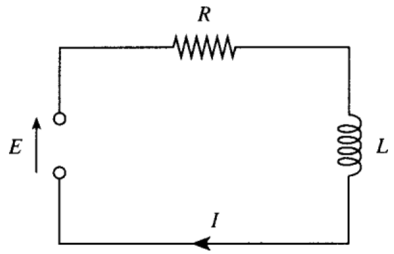

Figure 3-3

Electrical Circuits

The basic equation governing the amount of current *I* (in amperes) in a simple RL circuit (see Figure 3-3) consisting of a resistance *R* (in ohms), an inductor *L* (in henries), and an electromotive force (abbreviated emf) *E* (in volts) is

$$\frac{dI}{dt} + \frac{R}{L} I = \frac{E}{L} \tag{3.9}$$

For an RC circuit consisting of a resistance, a capacitance *C* (in farads), an emf, and no inductance (Figure 3-4), the equation governing the amount of electrical charge *q* (in coulombs) on the capacitor is

$$\frac{dq}{dt} + \frac{1}{RC} q = \frac{E}{R} \tag{3.10}$$

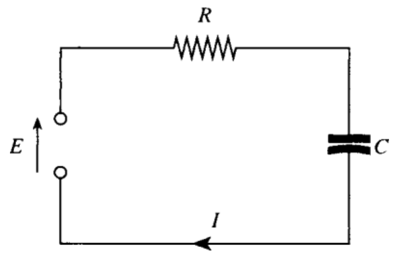

Figure 3-4

The relationship between q and I is

$$I = \frac{dq}{dt} \tag{3.11}$$

For more complex circuits see Chapter Seven.

Orthogonal Trajectories

Consider a one-parameter family of curves in the xy-plane defined by

$$F(x, y, c) = 0 \tag{3.12}$$

where c denotes the parameter. The problem is to find another one-parameter family of curves, called the *orthogonal trajectories* of the family of curves in 3.12 and given analytically by

$$G(x, y, k) = 0 \tag{3.13}$$

such that every curve in this new family 3.13 intersects at right angles every curve in the original family 3.12.

We first implicitly differentiate 3.12 with respect to x, then eliminate c between this derived equation and 3.12. This gives an equation connecting x, y, and y', which we solve for y' to obtain a differential equation of the form

$$\frac{dy}{dx} = f(x, y) \tag{3.14}$$

The orthogonal trajectories of 3.12 are the solutions of

$$\frac{dy}{dx} = -\frac{1}{f(x,y)} \tag{3.15}$$

For many families of curves, one cannot explicitly solve for dy / dx and obtain a differential equation of the form 3.14. We do not consider such curves in this book.