LECTUER (7) :
Orthogonal Trajectories

Consider a one-parameter family of curves in the *xy*-plane defined by

$$F(x, y, c) = 0 \tag{3.12}$$

where c denotes the parameter. The problem is to find another one-parameter family of curves, called the *orthogonal trajectories* of the family of curves in 3.12 and given analytically by

$$G(x, y, k) = 0 \tag{3.13}$$

such that every curve in this new family 3.13 intersects at right angles every curve in the original family 3.12.

We first implicitly differentiate 3.12 with respect to x, then eliminate c between this derived equation and 3.12. This gives an equation connecting x, y, and y', which we solve for y' to obtain a differential equation of the form

$$\frac{dy}{dx} = f(x, y) \tag{3.14}$$

The orthogonal trajectories of 3.12 are the solutions of

$$\frac{dy}{dx} = -\frac{1}{f(x,y)} \tag{3.15}$$

For many families of curves, one cannot explicitly solve for dy / dx and obtain a differential equation of the form 3.14. We do not consider such curves in this book.

Example: A bacteria culture is known to grow at a rate pro- portional to the amount present. After one hour, 1000 strands of the bac-

teria are observed in the culture; and after four hours, 3000 strands. Find (a) an expression for the approximate number of strands of the bacteria present in the culture at any time t and (b) the approximate number of strands of the bacteria originally in the culture.

(a) Let $N(t)$ denote the number of bacteria strands in the culture at time t. From Equation 3.1, $dN/dt - kN = 0$, which is both linear and separable. Its solution is

$$N(t) = ce^{kt} \tag{3.16}$$

At $t = 1$, $N = 1000$; hence,

$$1000 = ce^{k} \tag{3.17}$$

At $t = 4$, $\mathrm{N} = 3000$; hence.

$$3000 = ce^{4k} \tag{3.18}$$

Solving 3.17 and 3.18 for k and c, we find

$$k = \frac{1}{} \ln 3 \approx 0.3662 \text{ and } c = 1000e^{-k} = 693\ 3$$

Substituting these values of k and c into 3.16, we obtain

$$N(t) = 693e^{0.3662t} \qquad (3.19)$$

(b) *We* require N at $t = 0$. Substituting $t = 0$ into 3.19, we obtain $N(0) = 693e^{(0.3662)(0)} = 693$.

Solved Problem : A tank initially holds 100 gal of a brine solution containing 20 lb of salt. At $t = 0$, fresh water is poured into the tank at the rate of 5 gal/min, while the well-stirred mixture leaves the tank at the same rate. Find the amount of salt in the tank at any time t.

Here, $V_0 = 100$, $a = 20$, $b = 0$, and $e = f = 5$. Equation 3.8 becomes

The solution of this $\frac{dQ}{dt} + \frac{1}{20}Q = 0$

linear equation is

$$Q = ce^{-t/20} \qquad (3.20)$$

At $t = 0$, we are given that $Q = a = 20$. Substituting these values into 3.20, we find that $c = 20$, so that 3.20 can be rewritten as $Q = 20e^{-t/20}$. Note that a