LECTUER (8):

Chapter 4

Linear Differential Equations: Theory of Solutions

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Linear Differential Equations

An nth-order linear differential equation has the form

$$b_n(x)y^{(n)} + b_{n-1}(x)y^{(n-1)} + \cdots + b_1(x)y' + b_0(x)y = g(x) \tag{4.1}$$

where $g(x)$ and the coefficients $b_j(x)$ ($j = 0,1,2,\ldots, n$) depend solely on the variable x. In other words, they do *not* depend on y or any derivative of y.

If $g(x) = 0$, then Equation 4.1 is *homogeneous*; if not, 4.1 is *nonhomogeneous*. A linear differential equation has *constant coefficients* if all the coefficients $b_j(x)$ in 4.1 are constants; if one or more of these coefficients is not constant, 4.1 has *variable coefficients*.

Theorem 4.1. Consider the initial-value problem given by the linear differential equation 4.1 and the n initial conditions

$$\begin{gathered} y(x_0) = c_0, \quad y'(x_0) = c_1, \\ y''(x_0) = c_2, \ldots, y^{(n-1)}(x_0) = c_{n-1} \end{gathered} \tag{4.2}$$

If $g(x)$ and $b_j(x)$ ($j = 0,1,2,..., n$) are continuous in some interval P containing x_0 and if $b_n(x) \neq 0$ in P, then the initial-value problem given by 4.1 and 4.2 has a unique (only one) solution defined throughout P.

When the conditions on $b_n(x)$ in Theorem 4.1 hold, we can divide Equation 4.1 by $b_n(x)$ to get

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = f(x) \tag{4.3}$$

where $a_j(x) = b_j(x)/b_n(x)$ ($j = 0,1,2,..., n - 1$) and $f(x) = g(x)/b_n(x)$.

Let us define the differential operator $\mathbf{L}(y)$ by

$$\mathbf{L}(y) \equiv y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y \quad (4.4)$$

where $a_i(x)$ ($i = 0,1,2,..., n - 1$) is continuous on some interval of interest. Then 4.3 can be rewritten as

$$\mathbf{L}(y) = f(x) \tag{4.5}$$

and, in particular, a linear *homogeneous* differential equation can be expressed as

$$\mathbf{L}(y) = 0 \tag{4.6}$$

A set of functions $\{y_1(x), y_2(x),...,y_n(x)\}$ is *linearly dependent* on $a \le x \le b$ if there exist constants $c_1,c_2,...,c_n$ *not all zero*, such that

$$c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) \equiv 0 \tag{4.7}$$

on $a \le x \le b$.

Example 4.1: The set $\{x, 5x, 1, \sin x\}$ is linearly dependent on $[-1,1]$ since there exist constants $c_1 = -5$, $c_2 = 1$, $c_3 = 0$, and $c_4 = 0$, *not all zero*, such that 4.7 is satisfied. In particular,

$$-5 \cdot x + 1 \cdot 5x + 0 \cdot 1 + 0 \cdot \sin x = 0$$

Note that $c_1 = c_2 = \cdots c_n = 0$ is a set of constants that always satisfies 4.7. A set of functions is linearly dependent if there exists *another* set of constants, *not all zero*, that also satisfies 4.7. If the *only* solution to 4.7 is $c_1 = c_2 = \cdots c_n = 0$, then the set of functions $\{y_1(x), y_2(x),...,y_n(x)\}$ is *linearly independent* on $a \le x \le b$.

Theorem 4.2. The *nth*-order linear *homogeneous* differential equation $\mathbf{L}(y) = 0$ always has n linearly independent solutions. If $y_1(x), y_2(x),...,y_n(x)$ represent these solutions, then the general solution of $\mathbf{L}(y) = 0$ is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) \tag{4.8}$$

where $c_1,c_2,...,c_n$ denote arbitrary constants.

The Wronskian

The *Wronskian* of a set of functions $\{z_1(x), z_2(x),..., z_n(x)\}$ on the interval $a \le x \le b$, having the property that each function possesses $n - 1$ derivatives on this interval, is the determinant

$$W(z_1, z_2, \ldots, z_n) = \begin{vmatrix} z_1 & z_2 & \cdots & z_n \\ z_1' & z_2' & \cdots & z_n' \\ z_1'' & z_2'' & \cdots & z_n'' \\ \vdots & \vdots & \cdots & \vdots \\ z_1^{(n-1)} & z_2^{(n-1)} & \cdots & z_n^{(n-1)} \end{vmatrix}$$

Theorem 4.3. If the Wronskian of a set of n functions defined on the in- terval $a \le x \le b$ is nonzero for at least one point in this interval, then the set of functions is linearly independent there. If the Wronskian is identi- cally zero on this interval and if each of the functions is a solution to the same linear differential equation, then the set of functions is linearly de- pendent.

Caution: Theorem 4.3 is silent when the Wronskian is identically zero and the functions are not known to be solutions of the same linear differential equation. In this case, one must test directly whether Equa- tion 4.7 is satisfied.

Nonhomogeneous Equations

Let y_p denote any *particular* solution of Equation 4.5 (see Chapter One) and let y_h (henceforth called the *homogeneous* or *complementary solu- tion*) represent the *general* solution of the associated homogeneous equa- tion $\mathbf{L}(y) = 0$.

Theorem 4.4. The general solution to $\mathbf{L}(y) = f(x)$ is

$$y = y_h + y_p \tag{4.9}$$