Lecture (9): The Laplace transform

Reference: Boyce and DiPrima, Chapter 6

The Laplace transform is most useful for solving linear, constant-coefficient ode's when the inhomogeneous term or its derivative is discontinuous. Although ode's with discontinuous inhomogeneous terms can also be solved by adopting already learned methods, we will see that the Laplace transform technique provides a simpler, more elegant solution.

4.1 Definitions and properties of the forward and inverse Laplace transforms

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The main idea is to Laplace transform the constant-coefficient differential equation for $x(t)$ into a simpler algebraic equation for the Laplace-transformed function $X(s)$, solve this algebraic equation, and then transform $X(s)$ back into $x(t)$. The correct definition of the Laplace transform and the properties that this transform satisfies makes this solution method possible.

An exponential ansatz is used in solving homogeneous constant-coefficient odes, and the exponential function correspondingly plays a key role in defining the Laplace transform. The Laplace transform of $f(t)$, denoted by $F(s) = \mathcal{L}\{f(t)\}$, is defined by the integral transform

$$F(s) = \int_0^\infty e^{-st} f(t) dt. \qquad (4.1)$$

The improper integral given by (4.1) diverges if $f(t)$grows faster than e^{st} for large t. Accordingly, some restriction on the range of s may be required to guarantee convergence of (4.1), and we will assume without further elaboration that these restrictions are always satisfied.

The Laplace transform is a linear transformation. We have

$$\begin{aligned}
\mathcal{L}\{c_1 f_1(t) + c_2 f_2(t)\} &= \int_0^\infty e^{-st}\big(c_1 f_1(t) + c_2 f_2(t)\big) dt \\
&= c_1 \int_0^\infty e^{-st} f_1(t) dt + c_2 \int_0^\infty e^{-st} f_2(t) dt \\
&= c_1 \mathcal{L}\{f_1(t)\} + c_2 \mathcal{L}\{f_2(t)\}.
\end{aligned}$$

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. $e^{at} f(t)$	$F(s-a)$
2. 1	$\dfrac{1}{s}$
3. e^{at}	$\dfrac{1}{s-a}$
4. t^n	$\dfrac{n!}{s^{n+1}}$
5. $t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$
6. $\sin bt$	$\dfrac{b}{s^2+b^2}$
7. $\cos bt$	$\dfrac{s}{s^2+b^2}$
8. $e^{at}\sin bt$	$\dfrac{b}{(s-a)^2+b^2}$
9. $e^{at}\cos bt$	$\dfrac{s-a}{(s-a)^2+b^2}$
10. $t\sin bt$	$\dfrac{2bs}{(s^2+b^2)^2}$
11. $t\cos bt$	$\dfrac{s^2-b^2}{(s^2+b^2)^2}$
12. $u_c(t)$	$\dfrac{e^{-cs}}{s}$
13. $u_c(t) f(t-c)$	$e^{-cs} F(s)$
14. $\delta(t-c)$	e^{-cs}
15. $\dot{x}(t)$	$sX(s) - x(0)$
16. $\ddot{x}(t)$	$s^2 X(s) - sx(0) - \dot{x}(0)$

There is also a one-to-one correspondence between functions and their Laplace transforms. A table of Laplace transforms can therefore be constructed and used to find both Laplace and inverse Laplace transforms of commonly occurring functions. Such a table is shown in Table 4.1 (and this table will be distributed with the exams). In Table 4.1, n is a positive integer. Also, the cryptic entries for $u_c(t)$ and $\delta(t-c)$ will be explained later in §4.3.

The rows of Table 4.1 can be determined by a combination of direct integration and some tricks. We first compute directly the Laplace transform of $e^{at}f(t)$ (line 1):

$$\begin{aligned}\mathcal{L}\{e^{at}f(t)\} &= \int_0^\infty e^{-st}e^{at}f(t)dt \\ &= \int_0^\infty e^{-(s-a)t}f(t)dt \\ &= F(s-a).\end{aligned}$$

We also compute directly the Laplace transform of 1 (line 2):

$$\begin{aligned}\mathcal{L}\{1\} &= \int_0^\infty e^{-st}dt \\ &= -\frac{1}{s}e^{-st}\Big]_0^\infty \\ &= \frac{1}{s}.\end{aligned}$$

Now, the Laplace transform of e^{at} (line 3) may be found using these two results:

$$\begin{aligned}\mathcal{L}\{e^{at}\} &= \mathcal{L}\{e^{at}\cdot 1\} \\ &= \frac{1}{s-a}.\end{aligned} \tag{4.2}$$

The transform of t^n (line 4) can be found by successive integration by parts. A more interesting method uses Taylor series expansions for e^{at} and $1/(s-a)$. We have

$$\begin{aligned}\mathcal{L}\{e^{at}\} &= \mathcal{L}\left\{\sum_{n=0}^\infty \frac{(at)^n}{n!}\right\} \\ &= \sum_{n=0}^\infty \frac{a^n}{n!}\mathcal{L}\{t^n\}.\end{aligned} \tag{4.3}$$

Also, with $s > a$,

$$\begin{aligned}\frac{1}{s-a} &= \frac{1}{s(1-\frac{a}{s})} \\ &= \frac{1}{s}\sum_{n=0}^\infty \left(\frac{a}{s}\right)^n \\ &= \sum_{n=0}^\infty \frac{a^n}{s^{n+1}}.\end{aligned} \tag{4.4}$$

Using (4.2), and equating the coefficients of powers of a in (4.3) and (4.4), results in line 4:

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}.$$

The Laplace transform of $t^n e^{at}$ (line 5) can be found from line 1 and line 4:

$$\mathcal{L}\{t^n e^{at}\} = \frac{n!}{(s-a)^{n+1}}.$$

The Laplace transform of $\sin bt$ (line 6) may be found from the Laplace transform of e^{at} (line 3) using $a = ib$:

$$\begin{aligned}
\mathcal{L}\{\sin bt\} &= \mathrm{Im}\{\mathcal{L}\{e^{ibt}\}\} \\
&= \mathrm{Im}\left\{\frac{1}{s-ib}\right\} \\
&= \mathrm{Im}\left\{\frac{s+ib}{s^2+b^2}\right\} \\
&= \frac{b}{s^2+b^2}.
\end{aligned}$$

Similarly, the Laplace transform of $\cos bt$ (line 7) is

$$\begin{aligned}
\mathcal{L}\{\cos bt\} &= \mathrm{Re}\{\mathcal{L}\{e^{ibt}\}\} \\
&= \frac{s}{s^2+b^2}.
\end{aligned}$$

The transform of $e^{at}\sin bt$ (line 8) can be found from the transform of $\sin bt$ (line 6) and line 1:

$$\mathcal{L}\{e^{at}\sin bt\} = \frac{b}{(s-a)^2+b^2};$$

and similarly for the transform of $e^{at}\cos bt$:

$$\mathcal{L}\{e^{at}\cos bt\} = \frac{s-a}{(s-a)^2+b^2}.$$

The Laplace transform of $t\sin bt$ (line 10) can be found from the Laplace transform of te^{at} (line 5 with $n = 1$) using $a = ib$:

$$\begin{aligned}
\mathcal{L}\{t\sin bt\} &= \mathrm{Im}\{\mathcal{L}\{te^{ibt}\}\} \\
&= \mathrm{Im}\left\{\frac{1}{(s-ib)^2}\right\} \\
&= \mathrm{Im}\left\{\frac{(s+ib)^2}{(s^2+b^2)^2}\right\} \\
&= \frac{2bs}{(s^2+b^2)^2}.
\end{aligned}$$

Similarly, the Laplace transform of $t\cos bt$ (line 11) is

$$\begin{aligned}
\mathcal{L}\{t\cos bt\} &= \mathrm{Re}\{\mathcal{L}\{te^{ibt}\}\} \\
&= \mathrm{Re}\left\{\frac{(s+ib)^2}{(s^2+b^2)^2}\right\} \\
&= \frac{s^2-b^2}{(s^2+b^2)^2}.
\end{aligned}$$

We now transform the inhomogeneous constant-coefficient, second-order, linear inhomogeneous ode for $x = x(t)$,

$$a\ddot{x} + b\dot{x} + cx = g(t),$$

making use of the linearity of the Laplace transform:

$$a\mathcal{L}\{\ddot{x}\} + b\mathcal{L}\{\dot{x}\} + c\mathcal{L}\{x\} = \mathcal{L}\{g\}.$$

To determine the Laplace transform of $\dot{x}(t)$ (line 15) in terms of the Laplace transform of $x(t)$ and the initial conditions $x(0)$ and $\dot{x}(0)$, we define $X(s) = \mathcal{L}\{x(t)\}$, and integrate

$$\int_0^\infty e^{-st}\dot{x}dt$$

by parts. We let

$$\begin{aligned} u &= e^{-st} & dv &= \dot{x}dt \\ du &= -se^{-st}dt & v &= x. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_0^\infty e^{-st}\dot{x}dt &= xe^{-st}\Big]_0^\infty + s\int_0^\infty e^{-st}xdt \\ &= sX(s) - x(0), \end{aligned}$$

where assumed convergence of the Laplace transform requires

$$\lim_{t\to\infty} x(t)e^{-st} = 0.$$

Similarly, the Laplace transform of $\ddot{x}(t)$ (line 16) is determined by integrating

$$\int_0^\infty e^{-st}\ddot{x}dt$$

by parts and using the just derived result for the first derivative. We let

$$\begin{aligned} u &= e^{-st} & dv &= \ddot{x}dt \\ du &= -se^{-st}dt & v &= \dot{x}, \end{aligned}$$

so that

$$\begin{aligned} \int_0^\infty e^{-st}\ddot{x}dt &= \dot{x}e^{-st}\Big]_0^\infty + s\int_0^\infty e^{-st}\dot{x}dt \\ &= -\dot{x}(0) + s\big(sX(s) - x(0)\big) \\ &= s^2X(s) - sx(0) - \dot{x}(0), \end{aligned}$$

where similarly we assume $\lim_{t\to\infty} \dot{x}(t)e^{-st} = 0$.