Finding Roots of Functions

A root of a function is nothing more than a number for which the function is zero. In other words, finding the roots of a function, $g(x)$, is equivalent to solving

$$g(x) = 0$$

Example 3 Determine all the roots of $f(t) = 9t^3 - 18t^2 + 6t$

Solution

So we will need to solve,

$$9t^3 - 18t^2 + 6t = 0$$

First, we should factor the equation as much as possible. Doing this gives,

$$3t(3t^2 - 6t + 2) = 0$$

Next recall that if a product of two things are zero then one (or both) of them had to be zero. This means that,

$$3t = 0$$
$$\text{OR,}$$
$$3t^2 - 6t + 2$$

From the first it's clear that one of the roots must then be $t = 0$. To get the remaining roots we will need to use the quadratic formula on the second equation. Doing this gives,

$$\begin{aligned} t =& \frac{-(-6) \mp \sqrt{(-6)^2 - 4(3)(2)}}{2(3)} \\ =& 1 \mp \frac{1}{\sqrt{3}} \end{aligned}$$

The roots of this function are

$$t = 0$$
$$t = \frac{3 + \sqrt{3}}{3}$$
$$t = \frac{3 - \sqrt{3}}{3}$$

Function Composition

The composition of $f(x)$ and $g(x)$ is

$$(fog)(x) = f(g(x))$$

In other words, compositions are evaluated by plugging the second function listed into the first function listed. Note as well that order is important here. Interchanging the order will usually result in a different answer.

Example 4 Given $f(x) = 3x^2 - x + 10$ and $g(x) = 1 - 20x$ find each of the following.

(a) $(fog)(5)$

(b) $(fog)(x)$

(c) $(gof)(x)$

(d) $(gog)(x)$

Solution

(a) $(fog)(5)$

In this case we've got a number instead of an x but it works in exactly the same way.

$$\begin{aligned}(fog)(5) &= f(g(5)) \\ &= f(-99) = 29512\end{aligned}$$

(b) $(fog)(x)$

$$\begin{aligned}(fog)(x) &= f(g(x)) \\ &= f(1 - 20x) \\ &= 3(1 - 20x)^2 - (1 - 20x) + 10 \\ &= 3(1 - 40x + 400x^2) - 1 + 20x + 10 \\ &= 1200x^2 - 100x + 12\end{aligned}$$

(c) $(gof)(x)$

$$\begin{aligned}(gof)(x) &= g(f(x)) \\ &= g(3x^2 - x + 10) \\ &= 1 - 20(3x^2 - x + 10) \\ &= -60x^2 + 20x - 199\end{aligned}$$

(d) $(gog)(x)$

$$(gog)(x) = g(g(x))$$

$$= g(1 - 20x)$$
$$= 1 - 20(1 - 20x)$$
$$= 400x - 19$$

0.1 Inverse Functions

Given two one-to-one functions $f(x)$ and $g(x)$ if

$$(fog)(x) = x \text{ and } (gof)(x) = x$$

then we say that $f(x)$ and $g(x)$ are ***inverses*** of each other. More specifically we will say that $g(x)$ is the inverse of $f(x)$ and denote it by

$$g(x) = f^{-1}(x)$$

Likewise we could also say that $f(x)$ is the ***inverse*** of $g(x)$ and denote it by

$$f(x) = g^{-1}(x)$$

When dealing with inverse functions we've got to remember that

$$f^{-1}(x) \neq \frac{1}{f(x)}$$

Finding the Inverse of a Function

Given the function $f(x)$ we want to find the inverse function, $f^{-1}(x)$.

1. Replace $f(x)$ with y. This is done to make the rest of the process easier.

2. Replace every x with a y and replace every y with an x.

3. Solve the equation from Step 2 for y. This is the step where mistakes are most often made so be careful with this step.

4. Replace y with $f^{-1}(x)$. In other words, we've managed to find the inverse at this point!

5. Verify your work by checking that $(fof^{-1})(x) = x$ and $(f^{-1}of)(x) = x$ are both true. This work can sometimes be messy making it easy to make mistakes so again be careful.

Example 1 Given $f(x) = 3x - 2$ find $f^{-1}(x)$.

Solution

First, we'll first replace $f(x)$ with y.

$$y = 3x - 2$$

Next, replace all $x's$ with y and all $y's$ with x.

$$x = 3y - 2$$

Now, solve for y

$$x + 2 = 3y$$
$$\frac{1}{3}(x + 2) = y$$
$$\frac{x}{3} + \frac{2}{3} = y$$

Finally replace y with $f^{-1}(x)$

$$f^{-1}(x) = \frac{x}{3} + \frac{2}{3}$$

Now, we'll check that $(fof^{-1})(x) = x$ is true.

$$\begin{aligned}(fof^{-1})(x) &= f[f^{-1}(x)] \\ &= f[\tfrac{x}{3} + \tfrac{2}{3}] \\ &= 3(\tfrac{x}{3} + \tfrac{2}{3}) - 2 \\ &= x + 2 - 2 \\ &= x\end{aligned}$$

0.2 Trig Functions

First let's start with the six trig functions and how they relate to each other.

$$\begin{array}{cc} \cos(x) & \sin(x) \\ \tan(x) = \frac{\sin(x)}{\cos(x)} & \cot(x) = \frac{\cos(x)}{\sin(x)} = \frac{1}{\tan(x)} \\ \sec(x) = \frac{1}{\cos(x)} & \csc(x) = \frac{1}{\sin(x)} \end{array}$$

All the trig functions can be defined in terms of a right triangle.

From this right triangle we get the following definitions of the six trig functions.

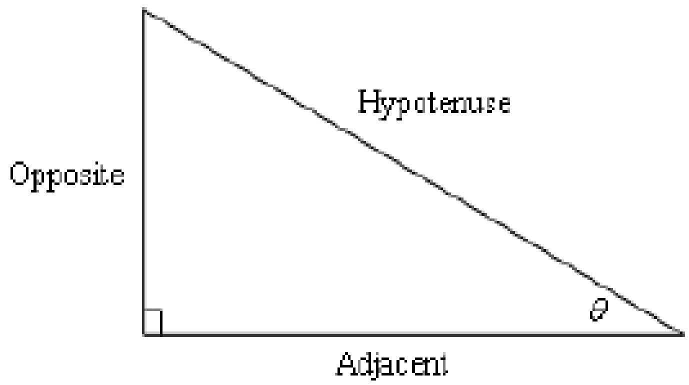

FIGURE 1

$$\begin{array}{cc} \cos\theta = \frac{adjacent}{hypotenuse} & \sin\theta = \frac{opposite}{hypotenuse} \\ \tan\theta = \frac{opposite}{adjacent} & \cot\theta = \frac{adjacent}{opposite} \\ \sec\theta = \frac{hypotenuse}{adjacent} & \csc\theta = \frac{hypotenuse}{opposite} \end{array}$$

The following table gives some of the basic angles in both degrees and radians.

Degree	0	30	45	60	90	180	270	360
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π

Odd and Even Trig Function

$$\begin{array}{cc} \sin(-\theta) = -\sin(\theta) & \text{Odd Function} \\ \cos(-\theta) = \cos(\theta) & \text{Even Function} \\ \tan(-\theta) = -\tan(\theta) & \text{Odd Function} \end{array}$$

The relationship among the trig function and its angle calculation can be shown in figure 2.

The unit circle is one of the more useful tools to come out of a trig function solution. See figure 3.

The way the unit circle works is to draw a line from the center of the circle outwards corresponding to a given angle. Then look at the coordinates of the point where the line and the circle intersect. The first coordinate is the cosine of that angle

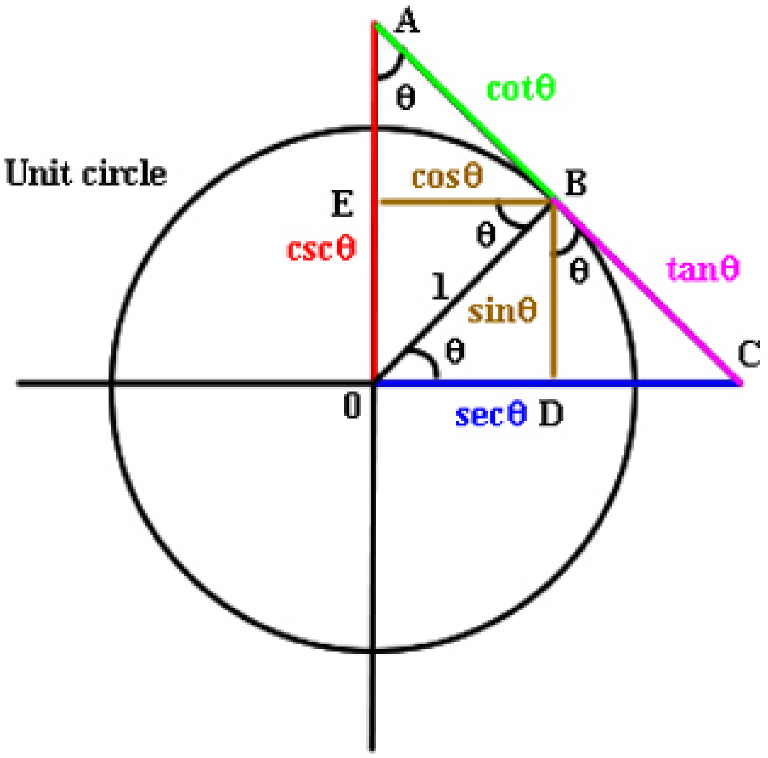

FIGURE 2

and the second coordinate is the sine of that angle. We've put some of the basic angles along with the coordinates of their intersections on the unit circle.

So, from the unit circle below we can see that $\cos(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$ and $\sin(\frac{\pi}{6}) = \frac{1}{2}$.

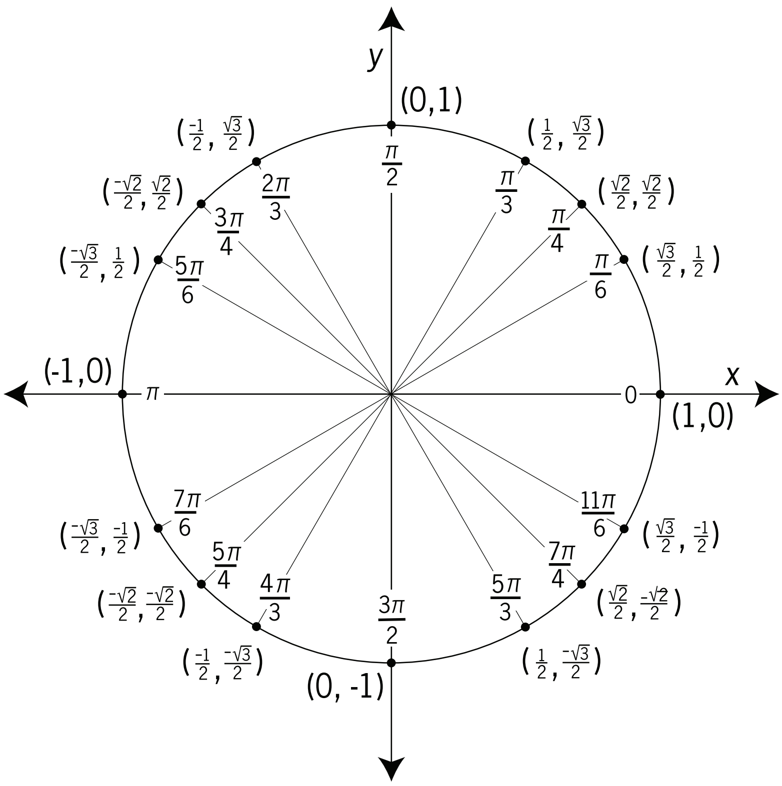
y
x
(0,1)
(1,0)
(-1,0)
(0, -1)
0
π/2
π
3π/2
π/6
π/4
π/3
2π/3
3π/4
5π/6
7π/6
5π/4
4π/3
5π/3
7π/4
11π/6
(1/2, √3/2)
(√2/2, √2/2)
(√3/2, 1/2)
(-1/2, √3/2)
(-√2/2, √2/2)
(-√3/2, 1/2)
(-√3/2, -1/2)
(-√2/2, -√2/2)
(-1/2, -√3/2)
(1/2, -√3/2)
(√2/2, -√2/2)
(√3/2, -1/2)

Figure 3