Example 1 Solve $\sin(2x) = -\cos(2x)$ on $[-\frac{3\pi}{2}, \frac{3\pi}{2}]$.

Solution

$$\begin{aligned} \sin(2x) &= -\cos(2x) \\ \tfrac{\sin(2x)}{\cos(2x)} &= -1 \\ \tan(2x) &= -1 \end{aligned}$$

Looking at our trusty unit circle it appears that the solutions will be,

$$\begin{aligned} 2x &= \tfrac{3\pi}{4} + 2\pi n, \qquad n = 0, \pm 1, \pm 2, \ldots \\ 2x &= \tfrac{7\pi}{4} + 2\pi n, \qquad n = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

Or, upon dividing by the 2 we get all possible solutions.

$$\begin{aligned} x &= \tfrac{3\pi}{8} + \pi n, \qquad n = 0, \pm 1, \pm 2, \ldots \\ x &= \tfrac{7\pi}{8} + \pi n, \qquad n = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

Now, let's determine the solutions that lie in the given interval.

$n = 0$.

$$\begin{aligned} x &= \tfrac{3\pi}{8} + \pi(0) = \tfrac{3\pi}{8} < \tfrac{3\pi}{2} \\ x &= \tfrac{7\pi}{8} + \pi(0) = \tfrac{7\pi}{8} < \tfrac{3\pi}{2} \end{aligned}$$

I'll leave the other $n's$ to you, to verify that $n = –3$ will give two answers that are both out of the interval.

The complete list of solutions is then,

$$-\tfrac{9\pi}{8}, -\tfrac{5\pi}{8}, -\tfrac{\pi}{8}, \tfrac{3\pi}{8}, \tfrac{7\pi}{8}, \tfrac{11\pi}{8}$$

Let's work one more example so that I can make a point that needs to be understood when solving some trig equations.

Example 2 Solve $\cos(3x) = 2$.

Solution

This example is designed to remind you of certain properties about sine and cosine. Recall that $-1 \leq \cos(\theta) \leq 1$ and $-1 \leq \sin(\theta) \leq 1$. Therefore, since cosine will never be greater that 1 it definitely can't be 2. So ***THERE ARE NO SOLUTIONS*** to this equation!

It is important to remember that not all trig equations will have solutions.

Inverse of Trig Functions

We'll only be looking at three of them and they are:

$$\begin{array}{rcrcl} InverseCosine & : & \cos^{-1}(x) & = & arccos(x) \\ InverseSine & : & \sin^{-1}(x) & = & arcsin(x) \\ InverseTangent & : & \tan^{-1}(x) & = & arctan(x) \end{array}$$

Example 1 Solve $-10\cos(3t) = 7$ on $[-2, 5]$.

Solution

First get the inverse cosine portion of this problem taken care of.

$$\cos(3t) = -\tfrac{7}{10} \qquad \Longrightarrow \qquad 3t = \cos^{-1}(\tfrac{7}{10}) = 2.3462$$

Now, let's look at a quick unit circle for this problem. As we can see the angle 2.3462 radians is in the second quadrant and the other angle that we need is in the third quadrant. We can find this second angle in:

We can use either -2.3462 or we can use $2\pi - 2.3462 = 3.9370$. You could use the negative if you wanted to.

So, let's now finish out the problem. First, let's acknowledge that the values of $3t$ that we need are,

$$\begin{array}{ll} 3t = 2.3462 + 2\pi n & n = 0, \pm 1, \pm 2, \ldots \\ 3t = 3.9370 + 2\pi n & n = 0, \pm 1, \pm 2, \ldots \end{array}$$

Now, we need to properly deal with the 3, so divide that out to get all the solutions to the trig equation.

$$\begin{array}{ll} t = 0.7821 + \frac{2\pi n}{3} & n = 0, \pm 1, \pm 2, \ldots \\ t = 1.3123 + \frac{2\pi n}{3} & n = 0, \pm 1, \pm 2, \ldots \end{array}$$

Finally, we need to get the values in the given interval.

$$\begin{array}{ccccc} n = -2 & : & t = -3.4067 & and & -2.8765 & \text{Both are Out of the Interval} \\ n = -1 & : & t = -1.3123 & and & -0.7821 & \text{In Interval} \\ n = 0 & : & t = 0.7821 & and & 1.3123 & \text{In Interval} \\ n = 1 & : & t = 2.8765 & and & 3.4067 & \text{In Interval} \\ n = 2 & : & t = 4.9709 & and & 5.5011 & \text{The Second one is Out of the Interval} \end{array}$$

The solutions to this equation, in the given interval are then,

$$t = -1.3123, -0.7821, 0.7821, 1.3123, 2.8765, 3.4067, 4.9709$$

0.1 Exponential and Logarithm Functions

Let's start with $b > 0$, $b \neq 1$. An exponential function is then a function in the form,

$$f(x) = b^x$$

Note that we avoid $b = 1$ because that would give the constant function, $f(x) = 1$. We avoid $b = 0$ since this would also give a constant function and we avoid negative values of b for the following reason.

Let's, for a second, suppose that we did allow b to be negative and look at the following function.

$$g(x) = (-4)^x$$

Let's do some evaluation.

$$g(2) = (-4)^2 = 16 \qquad g(\tfrac{1}{2}) = -(-4)^{\frac{1}{2}} = \sqrt{-4} = 2i$$

So, for some values of x we will get real numbers and for other values of x we well get complex numbers. We want to avoid this and so if we require $b > 0$ this will not be a problem.

Properties of $f(x) = b^x$

1. $f(0) = 1$. The function will always take the value of 1 at $x = 0$.

2. $f(x) \neq 0$. An exponential function will never be zero.

3. $f(x) > 0$. An exponential function is always positive.

4. The previous two properties can be summarized by saying that the range of an exponential function is $(0, \infty)$.

5. The domain of an exponential function is $(-\infty, \infty)$. In other words, you can plug every x into an exponential function.

6. If $0 < b < 1$ then,
   (a) $f(x) \to 0$ as $x \to \infty$
   (b) $f(x) \to \infty$ as $x \to -\infty$.

7. If $b > 1$ then,
   (a) $f(x) \to \infty$ as $x \to \infty$
   (b) $f(x) \to 0$ as $x \to -\infty$.

There is a very important exponential function that arises naturally in many places. This function is called the ***Natural Exponential Function***. However, for must people this is simply the exponential function.

Definition : The **natural exponential function** is $f(x) = \mathbf{e}^x$ where, $\mathbf{e} = 2.71828182845905\ldots$.

So, since $\mathbf{e} > 1$ we also know that $\mathbf{e}^x \to \infty$ as $x \to \infty$ and $\mathbf{e}^x \to 0$ as $x \to -\infty$.

Logarithm Function

We'll start with $b > 0$, $b \neq 1$ just as we did in the previous part. Then we have

$$y = \log_b x \qquad \text{is equivalent to} \qquad x = b^y$$

The first is called ***logarithmic form*** and the second is called the ***exponential form***. Remembering this equivalence is the key to evaluating logarithms. The number, b, is called the base.

Example 1 Without a calculator give the exact value of each of the following logarithms.

(a) $\log_2 16$

(b) $\log_4 16$

(c) $\log_5 625$

(d) $\log_9 \frac{1}{531441}$

(e) $\log_{\frac{1}{6}}$

(f) $\log_{\frac{3}{2}} \frac{27}{8}$

Solution

To quickly evaluate logarithms the easiest thing to do is to convert the logarithm to exponential form. So, let's take a look at the first one.

(a) $\log_2 16$

First, let's convert to exponential form.

$$\log_2 16 =? \qquad \text{is equivalent to} \qquad 2^? = 16$$

So, we're really asking 2 raised to what gives 16. Since 2 raised to 4 is 16 we get,

$$\log_2 16 = 4 \qquad \text{because} \qquad 2^4 = 16$$

(f) $\log_{\frac{3}{2}} \frac{27}{8}$

$$\log_{\frac{3}{2}} \frac{27}{8} = 3 \qquad \text{because} \qquad \frac{3}{2}^3 = \frac{27}{8}$$

There are a couple of special logarithms that arise in many places. These are,

$$\ln x = \log_{\mathbf{e}} x \qquad \text{This log is called the natural logarithm}$$
$$\log x = \log_1 0x \qquad \text{This log is called the common logarithm}$$

According to this,

$$\ln x \to \infty \qquad \text{as} \qquad x \to \infty$$
$$\ln x \to -\infty \qquad \text{as} \qquad x \to 0, x > 0$$

Example 2 Without a calculator give the exact value of each of the following logarithms.

(a) $\ln \sqrt[3]{\mathbf{e}}$

(b) $\log 1000$

(c) $\log_1 616$

(d) $\log_{23} 1$

(e) $\log_2 \sqrt[7]{32}$

Solution

(a)$\ln \sqrt[3]{\mathbf{e}} = \frac{1}{3}$ because $\mathbf{e}^{\frac{1}{3}} = \sqrt[3]{\mathbf{e}}$

(b)$\log 1000 = 3$ because $10^3 = 1000$

(c)$\log_1 616 = 1$ because $16^1 = 16$

(d)$\log_2 31 = 0$ because $23^0 = 1$

(e)$\log_2 \sqrt[7]{32} = \frac{5}{7}$ because $\sqrt[7]{32} = 32^{\frac{1}{7}} = (2^5)^{\frac{1}{7}} = 2^{\frac{5}{7}}$

Logarithm Properties

1. The domain of the logarithm function is $(0, \infty)$. In other words, we can only plug positive numbers into a logarithm! We can't plug in zero or a negative number.
2. $\log_b b = 1$.
3. $\log_b 1 = 0$.
4. $\log_b b^x = x$.
5. $b^{\log_b x} = x$.
6. According 4 and 5, $f(x) = b^x$ and$g(x) = \log_b x$are inverses of each other.
7. $\log_b xy = \log_b x + \log_b y$.
8. $\log_b(\frac{x}{y}) = \log_b x - \log_b y$.
9. $\log_b x^r = r \log_b x$.
10. $\log_b(x + y) \neq \log_b x + \log_b y$.
11. $\log_b(x - y) \neq \log_b x - \log_b y$.
12. $\log_b x = \frac{\ln x}{\ln b}$ and $\log_b x = \frac{\log x}{\log b}$.