Chapter 1

Limits and its Properties

1.1 Introduction

Consider the function $f(x) = \frac{x^2-1}{x-1}$

You can see that the function $f(x)$ is not defined at $x = 1$ as $x - 1$ is in the denominator. Take the value of x very nearly equal to but not equal to 1 as given in the tables below. In this case $x - 1 \neq 0$ as $x \neq 1$.

$\therefore$ We can write $f(x) = \frac{x^2-1}{x-1} = \frac{(x+1)(x-1)}{(x-1)} = x+1$, because $x-1 \neq 0$ and so division by $(x - 1)$ is possible.

In the above tables, you can see that as x gets closer to 1, the corresponding value of $f(x)$ also gets closer to 2.

x	$f(x)$
0.5	1.5
0.6	1.6
0.7	1.7
0.8	1.8
0.9	1.9
0.91	1.91
:	:
0.99	1.99
:	:
0.9999	1.9999

x	$f(x)$
1.9	2.9
1.8	2.8
1.7	2.7
1.6	2.6
1.5	2.5
:	:
1.01	2.01
1.001	2.001
:	:
1.00001	2.00001

Table 1.1

However, in this case $f(x)$ is not defined at $x = 1$. The idea can be expressed by saying that the limiting value of $f(x)$ is 2 when x approaches to 1.

Let us consider another function $f(x) = 2x$. Here, we are interested to see its behaviour near the point 1 and at $x = 1$. We find that as x gets nearer to 1, the corresponding value of $f(x)$ gets closer to 2 at $x = 1$ and the value of $f(x)$ is also 2.

So from the above findings, what more can we say about the behaviour of the function near $x = 2$ and at $x = 2$?

In this lesson we propose to study the behaviour of a function near and at a particular point where the function may or may not be defined.

1.2 Definition of Limits

In the introduction, we considered the function $f(x) = \frac{x^2-1}{x-1}$. We have seen that as x approaches l, $f(x)$ approaches 2. In general, if a function $f(x)$ approaches L when x approaches $'a'$, we say that L is the limiting value of $f(x)$.

Symbolically it is written as

Definition

We say that the limit of $f(x)$ is L as x approaches a and write this as

$$\lim_{x \to a} f(x) = L$$

provided we can make $f(x)$ as close to L as we want for all x sufficiently close to a, from both sides, without actually letting x be a.

Now let us find the limiting value of the function $(5x - 3)$ when x approaches 0.

$$\lim_{x \to 0}(5x - 3)$$

For finding this limit, we assign values to x from left and also from right of 0.

x	-0.1	-0.01	-0.001	$-.0001\ldots\ldots$
$5x-3$	-3.5	-3.05	-3.005	$-3.005\ldots\ldots$

x	0.1	0.01	0.001	.0001 ......
$5x-3$	2.5	2.95	2.995	2.9995 ......

It is clear from the above that the limit of $(5x-3)$ as $x \to 0$ is -3

$$\lim_{x \to 0}(5x-3) = -3$$

This is illustrated graphically in the figure 1.1.

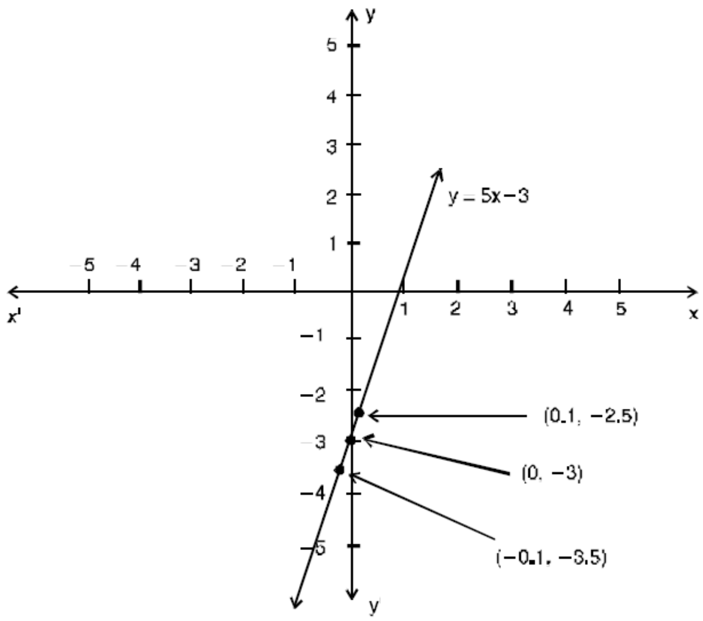

FIGURE 1.1

The method of finding limiting values of a function at a given point by putting the values of the variable very close to that point may not always be convenient.

We, therefore, need other methods for calculating the limits of a function as x (independent variable) ends to a finite quantity, say a

Example 1 Find $\lim_{x \to 3} f(x)$ where $f(x) = \frac{x^2-9}{x-3}$

Solution

We can solve it by the method of substitution. Steps of which are as follows:

Step 1: We consider a value of x close to a say $x = a + h$, where h is a very small positive number. Clearly, as $x \to a$, $h \to 0$.

For $f(x) = \frac{x^2-9}{x-3}$ we write $x = 3 + h$, so that as $x \to 3$, $h \to 0$.

Step 2: Simplify $f(x) = f(a+h)$.

Now,

$$\begin{aligned} f(x) &= f(3+h) \\ &= \frac{(3+h)^2 - 9}{3+h-3} \\ &= \frac{h^2+6h}{h} \\ &= h+6 \end{aligned}$$

Step 3: Put $h = 0$ and get the required result.

$$\lim_{x\to 3} f(x) = \lim_{h\to 0}(6+h)$$

As $x \to 0$, $h \to 0$

Thus,

$$\lim_{x\to 3} f(x) = 6+0 = 6$$

by putting $h = 0$.

Remarks : It may be noted that $f(3)$ is not defined, however, in this case the limit of the function $f(x)$ as $x \to 3$ is 6.

Now we shall discuss other methods of finding limits of different types of functions.

Example 2 Find $\lim_{x\to 1} f(x)$, where

$$f(x) = \begin{cases} \frac{x^3-1}{x^2-1}, & x \neq 1 \\ 1 & x = 1 \end{cases}$$

Solution

Here, for $x \neq 1$, $f(x) = \frac{x^3-1}{x^2-1}$

$$= \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)}$$

It shows that if $f(x)$ is of the form $\frac{g(x)}{h(x)}$, then we may be able to solve it by the method of factors. In such case, we follow the following steps :

Step 1:Factorise $g(x)$ and $h(x)$.

$$\begin{aligned} f(x) &= \frac{x^3-1}{x^2-1} \\ &= \frac{(x-1)(x^2+x+1)}{(x-1)(x+1)} \qquad \text{as} x \neq 1, x-1 \neq 0 \text{and such can be cancelled} \end{aligned}$$

Step 2: Simplify $f(x)$

$$f(x) = \frac{x^2+x+1}{x+1}$$

Step 3:Putting the value of x, we get the required limit.

$$\lim_{x\to 1} \frac{x^3-1}{x^2-1} = \frac{1+1+1}{1+1} = \frac{3}{2}$$

Thus, the limit of a function $f(x)$ as $x \to a$ may be different from the value of the function at $x = a$.

Now, we take an example which cannot be solved by the method of substitutions or method of factors.

Example 3 Evaluate

$$\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{x}$$

Solution

Here, we do the following steps:

Step 1:Rationalise the factor containing square root.

Step 2:Simplify

Step 3:Put the value of x and get the required result.

$$\begin{aligned}
\frac{\sqrt{1+x}-\sqrt{1-x}}{x} &= \frac{(\sqrt{1+x}-\sqrt{1-x})(\sqrt{1+x}+\sqrt{1-x})}{x(\sqrt{1+x}+\sqrt{1-x})}\\
&= \frac{\sqrt{(1+x)}^2-\sqrt{(1-x)}^2}{x(\sqrt{1+x}+\sqrt{1-x})}\\
&= \frac{(1+x)-(1-x)}{x(\sqrt{1+x}+\sqrt{1-x})}\\
&= \frac{1+x-1+x}{x(\sqrt{1+x}+\sqrt{1-x})}\\
&= \frac{2x}{x(\sqrt{1+x}+\sqrt{1-x})}, \qquad x \neq 0, \text{It can be cancelled}\\
&= \frac{2}{\sqrt{1+x}+\sqrt{1-x}}
\end{aligned}$$

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{x} &= \lim_{x\to 0}\frac{2}{\sqrt{1+x}+\sqrt{1-x}}\\
&= \frac{2}{\sqrt{1+0}+\sqrt{1-0}}\\
&= \frac{2}{1+1} = 1
\end{aligned}$$