Example 20.6 Evaluate : $\lim_{x\to 0}(|x|+|-x|)$

Solution : Since |x| has different values for $x \geq 0$ and x<0, therefore we have to find out both left hand and right hand limits.

$$\lim_{x\to 0^-}(|x|+|-x|) = \lim_{h\to 0}(|0-h| + |-(0-h)|)$$

$$= \lim_{h\to 0}(|-h|+|-(-h)|)$$

$$= \lim_{h\to 0} h + h = \lim_{h\to 0} 2h = 0 \qquad \text{...(i)}$$

and

$$\lim_{x\to 0^+}(|x|+|-x|) = \lim_{h\to 0}(|0+h| + |-(0+h)|)$$

$$= \lim_{x\to 0} h + h = \lim_{h\to 0} 2h = 0 \qquad \text{...(ii)}$$

From (i) and (ii),

$$\lim_{x\to 0^-}(|x|+|-x|) = \lim_{h\to 0^+}[|x| + |-x|]$$

Thus ,
$$\lim_{h\to 0}[|x|+|-x|] = 0$$

Note : We should remember that left hand and right hand limits are specially used when (a) the functions under consideration involve modulus function, and (b) function is defined by more than one rule.

Example 20.7 Find the vlaue of 'a' so that

$\lim_{x\to 1} f(x)$ exist, where $f(x)=\begin{cases}3x+5 & ,x\le 1\\ 2x+a, & x>1\end{cases}$

Solution : $\lim_{x\to 1^-} f(x)=\lim_{x\to 1}(3x+5)$ $\quad [\because f(x)=3x+5 \text{ for } x\le 1]$

$$=\lim_{h\to 0}[3(1-h)+5]$$

$$=3+5=8 \qquad .....(i)$$

$\lim_{x\to 1^+} f(x)=\lim_{x\to 1}(2x+a)$ $\quad [\because f(x)=2x+a \text{ for } x>1]$

$$=\lim_{h\to 0}(2(1+h)+a)$$

$$=2+a \qquad .....(ii)$$

We are given that $\lim_{x\to 1} f(x)$ will exists provided

$\Rightarrow$ $\lim_{x\to 1^-} = \lim_{x\to 1^+} f(x)$

$\therefore$ From (i) and (ii),

$2+a=8$

$\therefore$ or, $a=6$

Example 20.8 If a function f (x) is defined as

$$f(x)=\begin{cases}x & , \ 0\le x<\dfrac{1}{2}\\ 0 & , \ x=\dfrac{1}{2}\\ x-1, & \dfrac{1}{2}<x\le 1\end{cases}$$

Examine the existence of $\lim_{x\to\frac{1}{2}} f(x)$.

Solution : Here $f(x)=\begin{cases}x & , \ 0\le x<\dfrac{1}{2} & .....(i)\\ 0 & , \ x=\dfrac{1}{2}\\ x-1, & \dfrac{1}{2}<x\le 1 & .....(ii)\end{cases}$

$$\lim_{x\to\left(\frac{1}{2}\right)^-} f(x)=\lim_{h\to 0} f\left(\frac{1}{2}-h\right)$$

$$=\lim_{x\to 0}\left(\frac{1}{2}-h\right) \quad \left[\because \frac{1}{2}-h<\frac{1}{2} \text{ and from (i)}, f\left(\frac{1}{2}-h\right)=\frac{1}{2}-h\right]$$

$$= \frac{1}{2} - 0 = \frac{1}{2} \qquad \text{.....(iii)}$$

$$\lim_{x \to \left(\frac{1}{2}\right)^+} f(x) = \lim_{h \to 0} f\left(\frac{1}{2} + h\right)$$

$$= \lim_{h \to 0}\left[\left(\frac{1}{2} + h\right) - 1\right]\left[\because \frac{1}{2} + h > \frac{1}{2} \text{ and from(ii)}, f\left(\frac{1}{2} + h\right) = \left(\frac{1}{2} + h\right) - 1\right]$$

$$= \frac{1}{2} + -1$$

$$= -\frac{1}{2} \qquad \text{.....(iv)}$$

From (iii) and (iv), left hand limit $\neq$ right hand limit

$\therefore \quad \lim_{x \to \frac{1}{2}} f(x)$ does not exist.

CHECK YOUR PROGRESS 20.1

1. Evaluate each of the following limits :

(a) $\lim_{x \to 2}[2(x+3)+7]$ (b) $\lim_{x \to 0}\left(x^2+3x+7\right)$ (c) $\lim_{x \to 1}\left[(x+3)^2-16\right]$

(d) $\lim_{x \to \ 1}\left[(x+1)^2+2\right]$ (e) $\lim_{x \to 0}\left[(2x+1)^3-5\right]$ (f) $\lim_{x \to 1}(3x+1)(x+1)$

2. Find the limits of each of the following functions :

(a) $\lim_{x \to 5}\frac{x-5}{x+2}$ (b) $\lim_{x \to 1}\frac{x+2}{x+1}$ (c) $\lim_{x \to \ 1}\frac{3x+5}{x-10}$

(d) $\lim_{x \to 0}\frac{px+q}{ax+b}$ (e) $\lim_{x \to 3}\frac{x^2-9}{x-3}$ (f) $\lim_{x \to \ 5}\frac{x^2-25}{x+5}$

(g) $\lim_{x \to 2}\frac{x^2-x-2}{x^2-3x+2}$ (h) $\lim_{x \to \frac{1}{3}}\frac{9x^2-1}{3x-1}$

3. Evaluate each of the following limits:

(a) $\lim_{x \to 1}\frac{x^3-1}{x-1}$ (b) $\lim_{x \to 0}\frac{x^3+7x}{x^2+2x}$ (c) $\lim_{x \to 1}\frac{x^4-1}{x-1}$

(d) $\lim_{x \to 1}\left[\frac{1}{x-1} - \frac{2}{x^2-1}\right]$

4. Evaluate each of the following limits :

(a) $\lim_{x\to 0}\frac{\sqrt{4+x}-\sqrt{4-x}}{x}$ (b) $\lim_{x\to 0}\frac{\sqrt{2+x}-\sqrt{2}}{x}$ (c) $\lim_{x\to 3}\frac{\sqrt{3+x}-\sqrt{6}}{x-3}$

(d) $\lim_{x\to 0}\frac{x}{\sqrt{1+x}-1}$ (e) $\lim_{x\to 2}\frac{\sqrt{3x-2}-x}{2-\sqrt{6-x}}$

5. (a) Find $\lim_{x\to 0}\frac{2}{x}$, if it exists. (b) Find $\lim_{x\to 2}\frac{1}{x-2}$, if it exists.

6. Find the values of the limits given below :

(a) $\lim_{x\to 0}\frac{x}{5-|x|}$ (b) $\lim_{x\to 2}\frac{1}{|x+2|}$ (c) $\lim_{x\to 2}\frac{1}{|x-2|}$

(d) Show that $\lim_{x\to 5}\frac{|x-5|}{x-5}$ does not exist.

7. (a) Find the left hand and right hand limits of the function

$$f(x)=\begin{cases}-2x+3, & x\le 1\\ 3x-5, & x>1\end{cases} \text{ as } x\to 1$$

(b) If $f(x)=\begin{cases}x^2, & x\le 1\\ 1, & x>1\end{cases}$, find $\lim_{x\to 1}f(x)$

(c) Find $\lim_{x\to 4}f(x)$ if it exists, given that $f(x)=\begin{cases}4x+3, & x<4\\ 3x+7, & x\ge 4\end{cases}$

8. Find the value of 'a' such that $\lim_{x\to 2}f(x)$ exists, where $f(x)=\begin{cases}ax+5, & x<2\\ x-1, & x\ge 2\end{cases}$

9. Let $f(x)=\begin{cases}x, & x<1\\ 1, & x=1\\ x^2, & x>1\end{cases}$

Establish the existence of $\lim_{x\to 1}f(x)$.

10. Find $\lim_{x\to 2}f(x)$ if it exists, where

$$f(x)=\begin{cases}x-1, & x<2\\ 1, & x=2\\ x+1, & x>2\end{cases}$$

20.5 FINDING LIMITS OF SOME OF THE IMPORTANT FUNCTIONS

(i) Prove that $\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$ where n is a positive integer.

Proof : $\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a}=\lim\limits_{h\to 0}\dfrac{(a+h)^n-a^n}{a+h-a}$

$$=\lim_{h\to 0}\frac{\left(a^n+n\,a^{n-1}h+\dfrac{n(n-1)}{2!}a^{n-2}h^2+.....+h^n\right)-a^n}{h}$$

$$=\lim_{h\to 0}\frac{h\left(n\,a^{n-1}+\dfrac{n(n-1)}{2!}a^{n-2}h+.....+h^{n-1}\right)}{h}$$

$$=\lim_{h\to 0}\left[n\,a^{n-1}+\frac{n(n-1)}{2!}a^{n-2}h+.....+h^{n-1}\right]$$

$$=n\,a^{n-1}+0+0+.....+0$$

$$=n\,a^{n-1}$$

$\therefore$ $\quad\lim\limits_{x\to a}\dfrac{x^n-a^n}{x-a}=n\cdot a^{n-1}$

Note : However, the result is true for all n

(ii) Prove that (a) $\lim\limits_{x\to 0}\sin x=0$ and (b) $\lim\limits_{x\to 0}\cos x=1$

Proof : Consider a unit circle with centre B, in which $\angle C$ is a right angle and $\angle ABC$ = x radians.

Now sin x = A C and cos x = BC

As x decreases, A goes on coming nearer and nearer to C.

i.e., when $x\to 0, A\to C$

or when $x\to 0, AC\to 0$

and $BC\to AB$, i.e., $BC\to 1$

$\therefore$ When $x\to 0$ $\sin x\to 0$ and $\cos x\to 1$

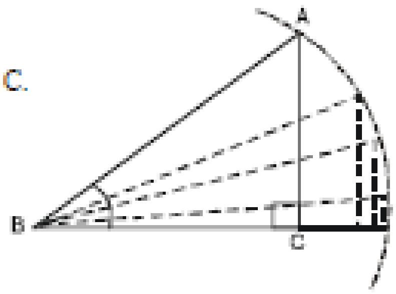

Fig. 20.3

Thus we have

$$\lim_{x\to 0}\sin x=0 \text{ and } \lim_{x\to 0}\cos x=1$$

(iii) Prove that $\lim\limits_{x\to 0}\dfrac{\sin x}{x}=1$

Proof : Draw a circle of radius 1 unit and with centre at the origin *O*. Let B (1,0) be a point on the circle. Let A be any other point on the circle. Draw $AC\perp OX$.

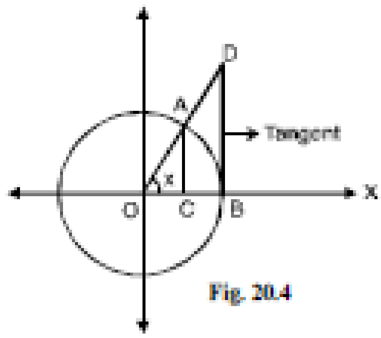

Fig. 20.4

Let $\angle AOX = x$ radians, where $0<x<\frac{\pi}{2}$

Draw a tangent to the circle at B meeting OA produced at D. Then $BD \perp OX$.

Area of $\Delta AOC <$ area of sector $OBA <$ area of ΔOBD.

or $\frac{1}{2} OC \times AC < \frac{1}{2} x(1)^2 < \frac{1}{2} OB \times BD$

$\left[\because \text{area of triangle} = \frac{1}{2}\text{base}\times\text{height and area of sector} = \frac{1}{2}\theta r^2\right]$

$\therefore \quad \frac{1}{2}\cos x \sin x < \frac{1}{2}x < \frac{1}{2}\cdot 1\cdot \tan x$

$\left[\because \cos x = \frac{OC}{OA}, \sin x = \frac{AC}{OA} \text{ and } \tan x = \frac{BD}{OB}, OA = 1 = OB\right]$

i.e., $\cos x < \frac{x}{\sin x} < \frac{\tan x}{\sin x}$ [Dividing throughout by $\frac{1}{2}\sin x$]

or $\cos x < \frac{x}{\sin x} < \frac{1}{\cos x}$

or $\frac{1}{\cos x} > \frac{\sin x}{x} < \cos x$

i.e., $\cos x < \frac{\sin x}{x} < \frac{1}{\cos x}$

Taking limit as $x \to 0$, we get

$$\lim_{x\to 0}\cos x < \lim_{x\to 0}\frac{\sin x}{x} < \lim_{x\to 0}\frac{1}{\cos x}$$

or $1 < \lim_{x\to 0}\frac{\sin x}{x} < 1$ $\left[\because \lim_{x\to 0}\cos x = 1 \text{ and } \lim_{x\to 0}\frac{1}{\cos x} = \frac{1}{1} = 1\right]$

Thus, $\lim_{x\to 0}\frac{\sin x}{x} = 1$

Note : In the above results, it should be kept in mind that the angle x must be expressed in radians.

(iv) Prove that $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$

Proof : By Binomial theorem, when $|x| < 1$, we get

$$(1+x)^{\frac{1}{x}}=\left[1+\frac{1}{x}x+\frac{\frac{1}{x}\left(\frac{1}{x}-1\right)}{2!}x^2++\frac{\frac{1}{x}\left(\frac{1}{x}-1\right)\left(\frac{1}{x}-2\right)}{3!}x^3+\ldots\ldots\infty\right]$$

$$=\left[1+1+\frac{(1-x)}{2!}+\frac{(1-x)(1-2x)}{3!}+\ldots\ldots\infty\right]$$

$$\therefore \lim_{x\to 0}(1+x)^{\frac{1}{x}}=\lim_{x\to 0}\left[1+1+\frac{1-x}{2!}+\frac{(1-x)(1-2x)}{3!}+\ldots\ldots\infty\right]$$

$$=\left[1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots\ldots\infty\right]$$

$$=e \quad \text{(By definition)}$$

Thus $\lim_{x\to 0}(1+x)^{\frac{1}{x}}=e$

(v) Prove that

$$\lim_{x\to 0}\frac{\log(1+x)}{x}=\lim_{x\to 0}\frac{1}{x}\log(1+x)=\lim_{x\to 0}\log(1+x)^{1/x}$$

$$=\log e \qquad \left(\text{Using } \lim_{x\to 0}(1+x)^{\frac{1}{x}}=e\right)$$

$$=1$$

(vi) Prove that $\lim_{x\to 0}\frac{e^x-1}{x}=1$

Proof : We know that $e^x=\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots\right)$

$$\therefore \qquad e^x-1=\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots-1\right)$$

$$=\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots\right)$$

$$\therefore \qquad \frac{e^x-1}{x}=\frac{\left(x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\ldots\right)}{x} \qquad \text{[Dividing throughout by x]}$$

$$=\frac{x\left(1+\frac{x}{2!}+\frac{x^2}{3!}+\ldots\ldots\right)}{x}$$

$$= \left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots\ldots\ldots\right)$$

$$\therefore \qquad \lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0}\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots\ldots\ldots\right)$$

$$= 1 + 0 + 0 + \ldots\ldots = 1$$

Thus, $\lim_{x \to 0} \frac{e^x - 1}{x} = 1$