**Example 20.11** Evaluate : $\lim_{x \to 0} \frac{\sin 3x}{x}$.

**Solution :** $\lim_{x \to 0} \frac{\sin 3x}{x} = \lim_{x \to 0} \frac{\sin 3x}{3x} \cdot 3$ [Multiplying and dividing by 3]

$= 3 \lim_{3x \to 0} \frac{\sin 3x}{3x}$ [$\because$ when $x \to 0, 3x \to 0$]

$= 3.1 \quad \left[\because \lim_{x \to 0} \frac{\sin x}{x} = 1\right]$

$= 3$

Thus, $\lim_{x \to 0} \frac{\sin 3x}{x} = 3$

**Example 20.12** Evaluate $\lim_{x \to 0} \frac{1 - \cos x}{2x^2}$.

**Solution :** $\lim_{x \to 0} \frac{1 - \cos x}{2x^2} = \lim_{x \to 0} \frac{2\sin^2 \frac{x}{2}}{2x^2}$ $\left[\begin{array}{l} \because \cos 2x = 1 - 2\sin^2 x, \\ \therefore 1 - \cos 2x = 2\sin^2 x \\ \text{or } 1 - \cos x = 2\sin^2 \frac{x}{2} \end{array}\right]$

$= \lim_{x \to 0} \left( \frac{\sin \frac{x}{2}}{2 \times \frac{x}{2}} \right)^2$ [Multiplying and dividing the denominator by 2]

$= \frac{1}{4} \lim_{\frac{x}{2} \to 0} \left( \frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2$

$$= \frac{1}{4} \times 1 = \frac{1}{4}$$

$$\therefore \qquad \lim_{x \to 0} \frac{1 - \cos x}{2x^2} = \frac{1}{4}$$

**Example 20.13** Evaluate : $\lim_{\theta \to 0} \frac{1 - \cos 4\theta}{1 - \cos 6\theta}$

**Solution :** $\lim_{\theta \to 0} \frac{1 - \cos 4\theta}{1 - \cos 6\theta} = \lim_{\theta \to \theta} \frac{2\sin^2 2\theta}{2\sin^2 3\theta}$

$$= \lim_{\theta \to 0} \left( \left( \frac{\sin 2\theta}{2\theta} \times 2\theta \right)^2 \left( \frac{3\theta}{\sin 3\theta} \times \frac{1}{3\theta} \right)^2 \right)$$

$$= \lim_{\theta \to 0} \left( \frac{\sin 2\theta}{2\theta} \right)^2 \left( \frac{3\theta}{\sin 3\theta} \right) \frac{4\theta^2}{9\theta^2}$$

$$= \left( \frac{4}{9} \right) \lim_{2\theta \to 0} \left( \frac{\sin 2\theta}{2\theta} \right)^2 \lim_{3\theta \to 0} \left( \frac{3\theta}{\sin 3\theta} \right)$$

$$= \frac{4}{9} \times 1 \times 1 = \frac{4}{9}$$

**Example 20.14** Find the value of $\lim_{x \to \frac{\pi}{2}} \frac{1 + \cos 2x}{(\pi - 2x)^2}$.

**Solution :** Put $x = \frac{\pi}{2} + h$ $\qquad \because$ when $x \to \frac{\pi}{2}, h \to 0$

$$\therefore \qquad 2x = \pi + 2h$$

$$\therefore \qquad \lim_{x \to \frac{\pi}{2}} \frac{1 + \cos 2x}{(\pi - 2x)^2} = \lim_{h \to 0} \frac{1 + \cos 2\left( \frac{\pi}{2} + h \right)}{[\pi - (\pi + 2h)]^2}$$

$$= \lim_{h \to 0} \frac{1 + \cos(\pi + 2h)}{4h^2}$$

$$= \lim_{h \to 0} \frac{1 - \cos 2h}{4h^2}$$

$$= \lim_{h \to 0} \frac{2\sin^2 h}{4h^2}$$

$$= \frac{1}{2} \lim_{h \to 0} \left( \frac{\sin h}{h} \right)^2$$

$$= \frac{1}{2} \times 1 = \frac{1}{2}$$

$$\therefore \qquad \lim_{x \to \frac{\pi}{2}} \frac{1 + \cos 2x}{(\pi - 2x)^2} = \frac{1}{2}$$

**Example 20.15** Evaluate $\lim_{x \to 0} \frac{\sin ax}{\tan bx}$

**Solution :** $\lim_{x \to 0} \frac{\sin ax}{\tan bx} = \lim_{x \to 0} \frac{\frac{\sin ax}{ax} \times a}{\frac{\tan bx}{bx} \times b}$

$$= \frac{a}{b} \frac{\lim_{x \to 0} \frac{\sin ax}{ax}}{\lim_{x \to 0} \frac{\tan bx}{bx}}$$

$$= \frac{a}{b} \cdot \frac{1}{1}$$

$$= \frac{a}{b}$$

$$\therefore \qquad \lim_{x \to 0} \frac{\sin ax}{\tan bx} = \frac{a}{b}$$

## CHECK YOUR PROGRESS 20.2

1. Evaluate each of the following :

(a) $\lim_{x \to 0} \frac{e^{2x} - 1}{x}$ (b) $\lim_{x \to 0} \frac{e^{x} - e^{-x}}{e^{x} + e^{-x}}$

2. Find the value of each of the following :

(a) $\lim_{x \to 1} \frac{e^{-x} - e^{-1}}{x - 1}$ (b) $\lim_{x \to 1} \frac{e - e^{x}}{x - 1}$

3. Evaluate the following :

(a) $\lim_{x \to 0} \frac{\sin 4x}{2x}$ (b) $\lim_{x \to 0} \frac{\sin x^2}{5x^2}$ (c) $\lim_{x \to 0} \frac{\sin x^2}{x}$

(d) $\lim_{x \to 0} \frac{\sin ax}{\sin bx}$

4. Evaluate each of the following :

(a) $\lim_{x\to 0} \frac{1-\cos x}{x^2}$ (b) $\lim_{x\to 0} \frac{1-\cos 8x}{x}$ (c) $\lim_{x\to 0} \frac{\sin 2x(1-\cos 2x)}{x^3}$

(d) $\lim_{x\to 0} \frac{1-\cos 2x}{3\tan^2 x}$

5. Find the values of the following :

(a) $\lim_{x\to 0} \frac{1-\cos ax}{1-\cos bx}$ (b) $\lim_{x\to 0} \frac{x^3 \cot x}{1-\cos x}$ (c) $\lim_{x\to 0} \frac{\operatorname{cosec} x - \cot x}{x}$

6. Evaluate each of the following :

(a) $\lim_{x\to} \frac{\sin x}{x\,\pi - x}$ (b) $\lim_{x\to 1} \frac{\cos \frac{\pi}{2} x}{1-x}$ (c) $\lim_{x\to \frac{\pi}{2}} (\sec x - \tan x)$

7. Evaluate the following :

(a) $\lim_{x\to 0} \frac{\sin 5x}{\tan 3x}$ (b) $\lim_{\theta\to 0} \frac{\tan 7\theta}{\sin 4\theta}$ (c) $\lim_{x\to 0} \frac{\sin 2x + \tan 3x}{4x - \tan 5x}$

## 20.5 CONTINUITY OF A FUNCTION AT A POINT

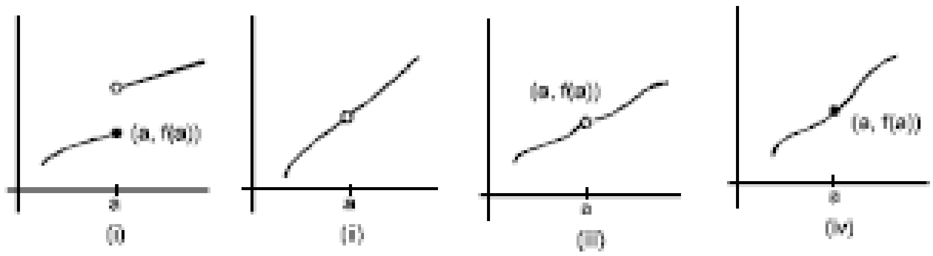


Fig. 20.5

Let us observe the above graphs of a function.

We can draw the graph (iv) without lifting the pencil but in case of graphs (i), (ii) and (iii), the pencil has to be lifted to draw the whole graph.

In case of (iv), we say that the function is continuous at x = a. In other three cases, the function is not continuous at x = a. i.e., they are discontinuous at x = a.

In case (i), the limit of the function does not exist at x = a.

In case (ii), the limit exists but the function is not defined at x = a.

In case (iii), the limit exists, but is not equal to value of the function at x = a.

In case (iv), the limit exists and is equal to value of the function at x = a.

**Example 20.16** Examine the continuity of the function f (x) = $x - a$ at x = a.

**Solution :** $\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$

$$= \lim_{h \to 0}[(a+h) - a]$$
$$= 0 \qquad \text{.....(i)}$$

Also $\qquad f(a) = a - a = 0 \qquad$ .....(ii)

From (i) and (ii),

$$\lim_{x \to a} f(x) = f(a)$$

Thus f (x) is continuous at x = a.

**Example 20.17** Show that f (x) = c is continuous.

**Solution :** The domain of constant function c is R.Let 'a' be any arbitrary real number.

$\therefore \qquad \lim_{x \to a} f(x) = c \text{ and } f(a) = c$

$\therefore \qquad \lim_{x \to a} f(x) = f(a)$

$\therefore$ f (x) is continuous at x = a. But 'a' is arbitrary. Hence f(x) = c is a constant function.

**Example 20.18** Show that $f(x) = cx + d$ is a continuous function.

**Solution :** The domain of linear function $f(x) = cx + d$ is R; and let 'a' be any arbitrary real number.

$$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$$
$$= \lim_{h \to a}[c\,(a+h) + d]$$
$$= ca + d \qquad \text{.....(i)}$$

Also $\qquad f(a) = ca + d \qquad$ .....(ii)

From (i) and (ii), $\lim_{x \to a} f(x) = f(a)$

$\therefore$ f (x) is continuous at x = a

and since a is any arbitrary , f (x) is a continuous function.

**Example 20.19** Prove that $f(x) = \sin x$ is a continuous function.

**Solution :** Let f (x) = sin x

The domain of sin x is R. let 'a' be any arbitrary real number.

$\therefore \qquad \lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$

$$= \lim_{h \to 0} \sin(a+h)$$

$$= \lim_{h \to 0} [\sin a.\cosh + \cos a.\sinh]$$

$$= \sin a \lim_{h \to 0} \cos h + \cos a \lim_{h \to 0} \sin h \qquad \left[\because \lim_{x \to a} kf(x) = k \lim_{x \to a} f(x) \text{ where k is a constant}\right]$$

$$= \sin a \times 1 + \cos a \times 0 \qquad \left[\because \lim_{x \to 0} \sin x = 0 \text{ and } \lim_{x \to 0} \cos x = 1\right]$$

$= \sin a$ .....(i)

Also $f(a) = \sin a$ .....(ii)

From (i) and (ii), $\lim_{x \to a} f(x) = f(a)$

$\therefore$ sin x is continuous at x = a

$\because$ sin x is continuous at x = a and 'a' is an aribitary point.

Therefore, $f(x) = \sin x$ is continuous.

**Example 20.20** Given that the function f (x) is continuous at x = 1. Find a when,

$f(x) = ax + 5$ and f (1) = 4.

**Solution :** $\lim_{x \to 1} f(x) = \lim_{h \to 0} (1 + h)$

$= \lim_{h \to 0} a(1+h) + 5$

$= a + 5$ .....(i)

Also f (1) = 4 .....(ii)

As f (x) is continuous at x = 1, therefore $\lim_{x \to 1} f(x) = f(1)$

$\therefore$ From (1) and (ii), $a + 5 = 4$

or $a = 4 - 5$ or $a = -1$