20.7 CONTINUITY OR OTHERWISE OF A FUNCTION AT A POINT

So far, we have considered only those functions which are continuous. Now we shall discuss some examples of functions which may or may not be continuous.

Example 20.22 Show that the function $f(x) = e^x$ is a continuous function.

Solution : Domain of e^x is R. Let $a \in R$, where 'a ' is arbitrary.

$$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a + h) \text{, where h is a very small number.}$$

$$= \lim_{h \to 0} e^{a+h}$$

$$= \lim_{h \to 0} e^a \cdot e^h$$

$$= e^a \lim_{h \to 0} e^h$$

$$= e^a \times 1 \qquad .....(i)$$

$$= e^a \qquad .....(ii)$$

Also $\qquad f(a) = e^a$

$\therefore$ From (i) and (ii), $\lim_{x \to a} f(x) = f(a)$

$\therefore$ f (x) is continuous at x = a

Since a is arbitary, e^x is a continuous function.

Example 20.23 By means of graph discuss the continuity of the function $f(x) = \dfrac{x^2 - 1}{x - 1}$.

Solution : The grah of the function is shown in the adjoining figure. The function is discontinuous as there is a gap in the graph at x = 1.

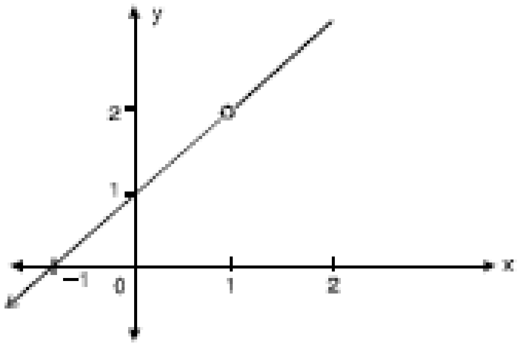

Fig. 20.6

CHECK YOUR PROGRESS 20.4

1. (a) Show that $f(x) = e^{5x}$ is a continuous function.

   (b) Show that $f(x) = e^{\frac{-2}{3}x}$ is a continuous function.

   (c) Show that $f(x) = e^{3x+2}$ is a continuous function.

   (d) Show that $f(x) = e^{-2x+5}$ is a continuous function.

2. By means of graph, examine the continuity of each of the following functions :

   (a) $f(x) = x+1$. (b) $f(x) = \dfrac{x+2}{x-2}$

   (c) $f(x) = \dfrac{x^2-9}{x+3}$ (d) $f(x) = \dfrac{x^2-16}{x-4}$

20.6 PROPERTIES OF CONTINUOUS FUNCTIONS

(i) Consider the function $f(x) = 4$. Graph of the function $f(x) = 4$ is shown in the Fig. 20.7. From the graph, we see that the function is continuous. In general, all constant functions are continuous.

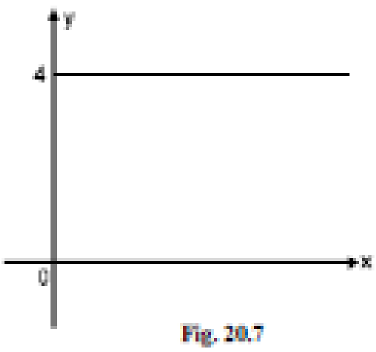

Fig. 20.7

(ii) If a function is continuous then the constant multiple of that function is also continuous.

Consider the function $f(x) = \frac{7}{2}x$. We know that x is a constant function. Let 'a' be an arbitrary real number.

$$\lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$$

$$= \lim_{h \to 0} \frac{7}{2}(a+h)$$

$$= \frac{7}{2}a \qquad \text{.....(i)}$$

Also $\qquad f(a) = \frac{7}{2}a \qquad$ .....(ii)

$\therefore$ From (i) and (ii),

$$\lim_{x \to a} f(x) = f(a)$$

$\therefore$ $f(x) = \frac{7}{2}x$ is continuous at $x = a$.

As $\frac{7}{2}$ is constant, and x is continuous function at $x = a$, $\frac{7}{2}x$ is also a continuous function at $x = a$.

(iii) Consider the function $f(x) = x^2 + 2x$. We know that the function x^2 and 2x are continuous.

Now $\qquad \lim_{x \to a} f(x) = \lim_{h \to 0} f(a+h)$

$$= \lim_{h \to 0}\left[(a+h)^2 + 2(a+h)\right]$$

$$= \lim_{h \to 0}\left[a^2 + 2ah + h^2 + 2a + 2ah\right]$$

$$= a^2 + 2a \qquad \text{.....(i)}$$

Also $\qquad f(a) = a^2 + 2a \qquad$ .....(ii)

$\therefore$ From (i) and (ii), $\lim_{x \to a} f(x) = f(a)$

$\therefore$ f(x) is continuous at $x = a$.

Thus we can say that if x^2 and 2x are two continuous functions at $x = a$ then $(x^2 + 2x)$ is also continuous at $x = a$.

(iv) Consider the function $f(x) = (x^2 + 1)(x + 2)$. We know that $(x^2 + 1)$ and $(x + 2)$ are two continuous functions.

Also $\qquad f(x) = (x^2 + 1)(x + 2)$

$$= x^3 + 2x^2 + x + 2$$

As $x^3, 2x^2, x$ and 2 are continuous functions, therefore,

$x^3 + 2x^2 + x + 2$ is also a continuous function.

$\therefore$ We can say that if $(x^2 + 1)$ and (x+2) are two continuous functions then $(x^2 + 1)(x + 2)$ is also a continuous function.

(v) Consider the function $f(x)=\frac{x^2-4}{x+2}$ at x = 2. We know that (x^2-4) is continuous at x = 2. Also (x + 2) is continuous at x = 2.

Again $$\lim_{x\to 2}\frac{x^2-4}{x+2}=\lim_{x\to 2}\frac{(x+2)(x-2)}{x+2}$$

$$=\lim_{x\to 2}(x-2)$$

$$=2-2=0$$

Also $$f(2)=\frac{(2)^2-4}{2+2}$$

$$=\frac{0}{4}=0$$

$\therefore$ $\lim_{x\to 2}f(x)=f(2)$. Thus f (x) is continuous at x = 2.

$\therefore$ If x^2-4 and $x+2$ are two continuous functions at x = 2, then $\frac{x^2-4}{x+2}$ is also continuous.

(vi) Consider the function $f(x)=|x-2|$. The function can be written as

$$f(x)=\begin{cases}-(x-2), x<2\\ (x-2), x\geq 2\end{cases}$$

$$\lim_{x\to 2^-}f(x)=\lim_{h\to 0}f(2-h), h>0$$

$$=\lim_{h\to 0}[(2-h)-2]$$

$$=2-2=0$$

$$\lim_{x\to 2^+}f(x)=\lim_{h\to 0}f(2+h),\ h>0 \quad \text{.....(i)}$$

$$=\lim_{x\to 2}[(2+h)-2]$$

$$=2-2=0 \quad \text{.....(ii)}$$

Also $$f(2)=(2-2)=0 \quad \text{.....(iii)}$$

$\therefore$ From (i), (ii) and (iii), $\lim_{x\to 2}f(x)=f(2)$

Thus, $|x-2|$ is continuous at x = 2.

After considering the above results, we state below some properties of continuous functions.

If f (x) and g (x) are two functions which are continuous at a point x = a, then

(i) C f (x) is continuous at x = a, where C is a constant.

(ii) $f(x)\pm g(x)$ is continuous at x = a.

(iii) $f(x)\cdot g(x)$ is continuous at x = a.

(iv) f (x)/g (x) is continuous at x = a, provided g (a) $\neq$ 0.

(v) |f(x)| is continuous at x = a.

Note : Every constant function is continuous.

20.9 IMPORTANT RESULTS ON CONTINUITY

By using the properties mentioned above, we shall now discuss some important results on continuity.

(i) Consider the function $f(x) = px + q, x \in R$. (i)

The domain of this functions is the set of real numbers. Let a be any arbitary real number. Taking limit of both sides of (i), we have

$$\lim_{x \to a} f(x) = \lim_{x \to a} (px + q) = pa + q \quad [= \text{value of } px + q \text{ at } x = a.]$$

$\therefore$ px +q is continuous at x = a.

Similarly, if we consider $f(x) = 5x^2 + 2x + 3$, we can show that it is a continuous function.

In general $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_{n-1}x^{n-1} + a_nx^n$

where $a_0, a_1, a_2 \dots a_n$ are constants and n is a non-negative integer,

we can show that $a_0, a_1x, a_2x^2, \dots a_nx^n$ are all continuos at a point x = c (where c is any real number) and by property (ii), their sum is also continuous at x = c.

$\therefore$ f (x) is continuous at any point c.

Hence every polynomial function is continuous at every point.

(ii) Consider a function $f(x) = \dfrac{(x+1)(x+3)}{(x-5)}$, $f(x)$ is not defined when $x - 5 = 0$ i.e. at x = 5.

Since (x + 1) and (x + 3) are both continuous, we can say that (x + 1) (x + 3) is also continuous. [Using property iii]

$\therefore$ Denominator of the function f (x), i.e., $(x - 5)$ is also continuous.

$\therefore$ Using the property (iv), we can say that the function $\dfrac{(x+1)(x+3)}{(x-5)}$ is continuous at all points except at x = 5.

In general if $f(x) = \dfrac{p(x)}{q(x)}$, where p (x) and q (x) are polynomial functions and q (x) $\neq$ 0, then f (x) is continuous if p (x) and q (x) both are continuous.

Example 20.24 Examine the continuity of the following function at x = 2.

$$f(x) = \begin{cases} 3x - 2 & \text{for } x < 2 \\ x + 2 & \text{for } x \geq 2 \end{cases}$$

Solution : Since f (x) is defined as the polynomial function $3x-2$ on the left hand side of the point x = 2 and by another polynomial function x +2 on the right hand side of x = 2, we shall find the left hand limit and right hand limit of the function at x = 2 separately.

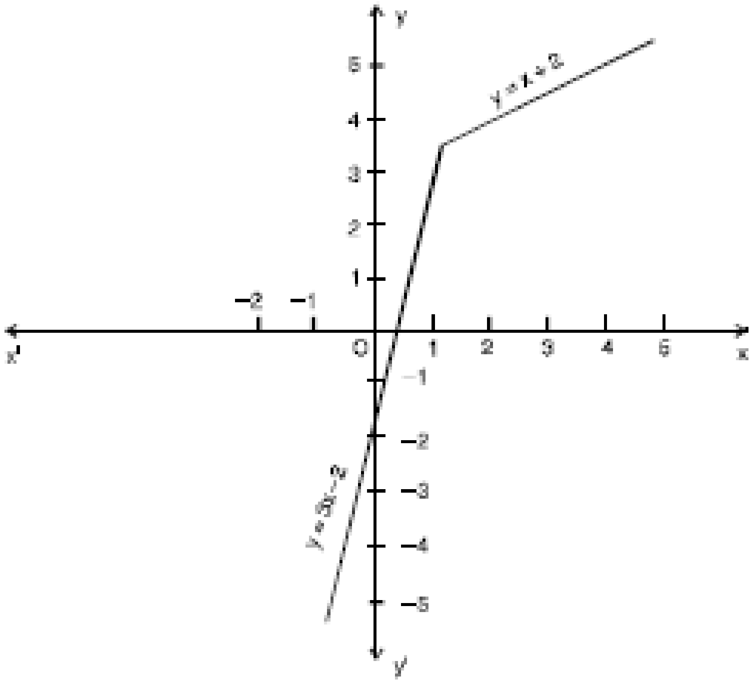

Fig. 20.8

Left hand limit $= \lim_{x \to 2^-} f(x)$

$= \lim_{x \to 2} (3x - 2)$

$= 3 \times 2 - 2 = 4$

Right hand limit at x = 2;

$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2} (x + 2) = 4$$

Since the left hand limit and the right hand limit at x = 2 are equal, the limit of the function f (x) exists at x =2 and is equal to 4 i.e., $\lim_{x \to 2} f(x) = 4$.

Also f (x) is defined by (x +2) at x = 2

$\therefore$ f (2) = 2 + 2 = 4.

Thus, $\lim_{x \to 2} f(x) = f(2)$

Hence f (x) is continuous at x = 2.