The Karnaugh Map (K-Map):

The Karnaugh map provides a systematic method for simplifying Boolean expression and, if properly used, will produce the simplest SOP or POS expression possible. The K-map, like a truth table, is a means for showing the relationship between logic inputs and the desired output. The K-map is an array of **Cells** in which each cell represents a binary value of the input variables. The cells are arranged in a way so that simplification of a given expression is simply a matter of properly grouping the cells. The number of cells in a Karnaugh map is equal to the total number of possible input variable combinations as is the number of rows in a truth table. For three variables, the number of cells is $2^3=8$. For four variables, the number of cells is $2^4=16$.

Three examples of K-maps for two, three, and four variables, together with the corresponding truth tables are shown in Figrue below:

A	**B**	**X**
0	0	1
0	1	0
1	0	0
1	1	1

$\overline{A}\,\overline{B}$

AB

$$\boxed{X=\overline{A}\,\overline{B}+AB}$$

	$\overline{B}$	B
$\overline{A}$	1	0
A	0	1

A	**B**	**C**	**X**
0	0	0	1

$\longrightarrow \overline{A}\,\overline{B}\,\overline{C}$

$\longrightarrow \overline{A}\,\overline{B}C$

$\longrightarrow \overline{A}B\overline{C}$

0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	0

	$\overline{C}$	C
$\overline{A}\overline{B}$	1	1
$\overline{A}B$	1	0
AB	1	0
$A\overline{B}$	0	0

A	**B**	**C**	**D**	**X**	
0	0	0	0	0	
0	0	0	1	1	$\overline{A}\overline{B}\overline{C}D$
0	0	1	0	0	
0	0	1	1	0	
0	1	0	0	0	
0	1	0	1	1	$\overline{A}B\overline{C}D$
0	1	1	0	0	
0	1	1	1	0	
1	0	0	0	0	
1	0	0	1	0	
1	0	1	0	0	
1	0	1	1	0	
1	1	0	0	0	
1	1	0	1	1	$AB\overline{C}D$
1	1	1	0	0	
1	1	1	1	1	$ABCD$

	$\overline{C}\overline{D}$	$\overline{C}D$	CD	$C\overline{D}$
$\overline{A}\overline{B}$	0	1	0	0
$\overline{A}B$	0	1	0	0
AB	0	1	1	0
$A\overline{B}$	0	0	0	0

$$\boxed{\begin{aligned} X &= \overline{A}\overline{B}\overline{C}D + \overline{A}B\overline{C}D \\ &\quad + AB\overline{C}D + ABCD \end{aligned}}$$

Karnaugh Map SOP Minimization:

As stated in the last section, the Karnaugh map is used for simplifying Boolean expressions to their minimum form. A minimized SOP expression contains the fewest possible terms with the fewest possible variables per term. Generally, a minimum SOP expression can be implemented with fewer logic gates than a standard expression and this is the basic purpose in the simplification process.

Example: Map the following SOP expression on a Karnaugh map:

$$\bar{A}\bar{B}C + \bar{A}B\bar{C} + AB\bar{C} + ABC$$

$\bar{A}\bar{B}C \longrightarrow$ **001**
$\bar{A}B\bar{C} \longrightarrow$ **010**
$AB\bar{C} \longrightarrow$ **110**
$ABC \longrightarrow$ **111**

	$\bar{C}$	C
$\bar{A}\bar{B}$		**1**
$\bar{A}B$	**1**	
AB	**1**	**1**
$A\bar{B}$		

Example: Map the following SOP expression on a Karnaugh map:

$$\bar{A} + A\bar{B} + AB\bar{C}$$

$\bar{A} \longrightarrow$
- $\bar{A}\bar{B}\bar{C}$ **000**
- $\bar{A}\bar{B}C$ **001**
- $\bar{A}B\bar{C}$ **010**
- $\bar{A}BC$ **011**

$A\bar{B} \longrightarrow$
- $A\bar{B}\bar{C}$ **100**
- $A\bar{B}C$ **101**

$AB\bar{C} \longrightarrow$ $AB\bar{C}$ **110**

	$\bar{C}$	C
$\bar{A}\bar{B}$	**1**	**1**
$\bar{A}B$	**1**	
AB	**1**	**1**
$A\bar{B}$	**1**	**1**

Looping:

The expression for output X can be simplified by properly combining these squares in the K-map which contains 1's. The process for combining these 1's is called **Looping**. Looping a pair of adjacent 1's in the K-map eliminates the variable that appears in complemented and un-complemented form as shown in the examples below:-

	$\bar{C}$	C
$\bar{A}\bar{B}$	0	0
$\bar{A}B$	**1**	0
AB	**1**	0
$A\bar{B}$	0	0

$$X = \bar{A}B\bar{C} + AB\bar{C}$$
$$= B\bar{C}$$

	$\bar{C}$	C
$\bar{A}\bar{B}$	0	0
$\bar{A}B$	**1**	**1**
AB	0	0
$A\bar{B}$	0	0

$$X = \bar{A}B\bar{C} + \bar{A}BC$$
$$= \bar{A}B$$

	$\bar{C}$	C
$\bar{A}\bar{B}$	**1**	0
$\bar{A}B$	0	0
AB	0	0
$A\bar{B}$	**1**	0

$$X = \bar{A}\bar{B}\bar{C} + A\bar{B}\bar{C}$$
$$= \bar{B}\bar{C}$$

	$\bar{C}\bar{D}$	$\bar{C}D$	CD	$C\bar{D}$
$\bar{A}\bar{B}$	0	0	**1**	**1**
$\bar{A}B$	0	0	0	0
AB	0	0	0	0
$A\bar{B}$	**1**	0	0	**1**

$\bar{A}\bar{B}C$

$A\bar{B}\bar{D}$

$$X = \bar{A}\bar{B}CD + \bar{A}\bar{B}C\bar{D} + A\bar{B}\bar{C}\bar{D} + A\bar{B}C\bar{D}$$
$$= \bar{A}\bar{B}C + A\bar{B}\bar{D}$$

The K-map may contain a group of four 1's that are adjacent to each other. This group is called a ***quad***. The simplification of such groups is shown in the examples below:

	$\overline{C}$	C
$\overline{A}\,\overline{B}$	0	**1**
$\overline{A}B$	0	**1**
AB	0	**1**
$A\overline{B}$	0	**1**

$X = C$

	$\overline{C}\,\overline{D}$	$\overline{C}D$	CD	$C\overline{D}$
$\overline{A}\,\overline{B}$	0	0	0	0
$\overline{A}B$	0	0	0	0
AB	**1**	**1**	**1**	**1**
$A\overline{B}$	0	0	0	0

$X = AB$

	$\overline{C}\,\overline{D}$	$\overline{C}D$	CD	$C\overline{D}$
$\overline{A}\,\overline{B}$	0	0	0	0
$\overline{A}B$	0	**1**	**1**	0
AB	0	**1**	**1**	0
$A\overline{B}$	0	0	0	0

$X = BD$

	$\overline{C}\,\overline{D}$	$\overline{C}D$	CD	$C\overline{D}$
$\overline{A}\,\overline{B}$	0	0	0	0
$\overline{A}B$	0	0	0	0
AB	**1**	0	0	**1**
$A\overline{B}$	**1**	0	0	**1**

$X = A\overline{D}$

	$\overline{C}\,\overline{D}$	$\overline{C}D$	CD	$C\overline{D}$
$\overline{A}\,\overline{B}$	**1**	0	0	**1**
$\overline{A}B$	0	0	0	0
AB	0	0	0	0
$A\overline{B}$	**1**	0	0	**1**

$X = \overline{B}\,\overline{D}$

Example: Simplify the following equation using Boolean algebra and Karnaugh map: $X = B.(A+\overline{B}).(B+C)$

1- **Using Boolean algebra**

$$\begin{aligned} X &= B.(A+\overline{B}).(B+C) \\ &= (AB+0).(B+C) \\ &= AB(B+C) \\ &= AB + ABC \\ &= AB(1+C) \\ &= AB \end{aligned}$$

2- **Using Karnaugh map**

	$\overline{C}$	C
$\overline{A}\,\overline{B}$		
$\overline{A}B$		
AB	**1**	**1**
$A\overline{B}$		

$X = AB + ABC$

$But\ AB = ABC + AB\overline{C}$

$X = ABC + AB\overline{C} + ABC$

$\quad = ABC + AB\overline{C}$

Example: Use a Karnaugh map to minimize the SOP expression:-

$$A\overline{B}C + \overline{A}BC + \overline{A}\,\overline{B}C + \overline{A}\,\overline{B}\,\overline{C} + A\overline{B}\,\overline{C}$$

$X = \overline{B} + \overline{A}C$

	$\overline{C}$	C
$\overline{A}\,\overline{B}$	**1**	**1**
$\overline{A}B$		**1**
AB		
$A\overline{B}$	**1**	**1**

$\overline{A}C$

$\overline{B}$

Example: Use a Karnaugh map to minimize the SOP expression:-

$$\overline{B}\,\overline{C}\,\overline{D} + \overline{A}B\overline{C}\,\overline{D} + AB\overline{C}\,\overline{D} + \overline{A}\,\overline{B}CD + A\overline{B}CD + \overline{A}\,\overline{B}C\overline{D} + \overline{A}BC\overline{D} + ABC\overline{D} + A\overline{B}C\overline{D}.$$

The first term $\overline{B}\,\overline{C}\,\overline{D}$ must be expanded into $A\overline{B}\,\overline{C}\,\overline{D}$ *and* $\overline{A}\,\overline{B}\,\overline{C}\,\overline{D}$ to get a standard SOP expression which is then mapped and the cells are grouped as shown in figure below:

	$\overline{C}\,\overline{D}$	$\overline{C}D$	CD	$C\overline{D}$
$\overline{A}\,\overline{B}$	**1**		**1**	**1**

$X = \bar{D} + \bar{B}C$

$\bar{A}B$	1			1
AB	1			1
$A\bar{B}$	1		1	1

$\bar{D}$ $\bar{B}C$

Don't Care Condition:

Some logic circuit can be designed so that there are certain input conditions for which there are no specified output levels, usually because these conditions will never occur. In other words, there will be certain combinations of input levels where we "don't care" whether the output is **HIGH** or **LOW**. This is illustrated in the truth table below:-

A	B	C	Z
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	x
1	0	0	x
1	0	1	1
1	1	0	1
1	1	1	1

Don't Care

	$\bar{C}$	C
$\bar{A}\bar{B}$	0	0
$\bar{A}B$	0	x
AB	**1**	**1**
$A\bar{B}$	x	**1**

	$\bar{C}$	C
$\bar{A}\bar{B}$	0	0
$\bar{A}B$	0	0
AB	**1**	**1**
$A\bar{B}$	**1**	**1**

$\boxed{Z = A}$

Whenever "don't care" conditions occur, we have to decide which ones to change to 0 and which to 1 to produce the best K-map looping (i.e. the simplest expression).

Karnaugh Map POS Minimization

In this section, we will focus on POS expressions instead of SOP. The approaches are much the same except that with POS expressions, 0's representing the standard sum terms are placed on the karnaugh map instead of 1's.

Example: Map the following POS expression on a karnaugh map.

$$(\bar{A}+\bar{B}+C+D)(\bar{A}+B+\bar{C}+\bar{D})(A+B+\bar{C}+D)(\bar{A}+\bar{B}+\bar{C}+\bar{D})(A+B+\bar{C}+\bar{D})$$

1100 1011 0010 1111 0011

	AB \ CD	00 $\bar{C}\bar{D}$	01 $\bar{C}D$	11 CD	10 $C\bar{D}$
00	$\bar{A}\bar{B}$			**0**	**0**
01	$\bar{A}B$				
11	AB	**0**		**0**	
10	$A\bar{B}$			**0**	

$\bar{A}+\bar{B}+\bar{C}+\bar{D}$

$\bar{A}+B+\bar{C}+\bar{D}$

Example: Use a karnaugh map to minimize the POS expression:

$$(A+B+C)(A+B+\bar{C})(A+\bar{B}+C)(A+\bar{B}+\bar{C})(\bar{A}+\bar{B}+C)$$

$Out = A(\bar{B}+C)$

keep in mind that this minimum POS expression is equivalent to the original standard POS expression grouping the 1's as shown yields a SOP expression that is equivalent to grouping the 0's .

	$\bar{C}$	C
$\bar{A}\bar{B}$	**0**	**0**
$\bar{A}B$	**0**	**0**
AB	**0**	**1**
$A\bar{B}$	**1**	**1**

A

AC

$$\boxed{AC + A\bar{B} = A(\bar{B}+C)}$$

Review Questions:

1. Use a Karnaugh map to minimize the POS expression: