## Chapter Three

## The Yield Criteria

### 3.1 Introduction

Suppose that an element of material is subjected to a system of stresses of gradually
increasing magnitude. The initial deformation of the element is entirely elastic and the original shape of the element is recovered on complete unloading. For certain critical combinations of the applied stresses, plastic deformation first appears in the element. A law defining the limit of elastic behavior under any possible combination of stresses is called yield criterion.
In developing a mathematical theory, it is necessary to take into account a number of idealizations at the outset. Firstly, it is assumed that the conditions of loading are such that all strain rate and thermal effects can be neglected. Secondly, the Bauschinger effect and the hysteresis loop, which arise from nonuniformity on the microscope scale, are disregarded. Finally, the material is assumed to be isotropic, so that its properties at each point are the same in all directions.

There is a useful and immediate simplification resulting from the experimental fact that yielding is practically unaffected by a uniform hydrostatic tension or compression.

### 3.2 Geometrical representation

Consider a system of three mutually perpendicular axes with the principal stresses taken as rectangular coordinates (Figure). The state of stress at any point in a body
may be represented by a vector emanating from the origin $O$. Imagine a line OH equally inclined to the three axes, so that its direction cosines are $(1 / \sqrt{ } 3$, $1 / \sqrt{ } 3,1 / \sqrt{ } 3)$.


The stress vector OQ, whose components are $\left(\sigma_{1}, \sigma_{2}, \sigma_{3}\right)$, may be resolved into a vector OG along OH and a vector OP perpendicular to OH . The vector OG is of magnitude: $(1 / \sqrt{ } 3)\left(\sigma_{1+} \sigma_{2+} \sigma_{3}\right)$ or $\sqrt{ } 3 \sigma_{m}$ and represents the hydrostatic stress.

The vector OP represents the deviatoric stress and:

$$
\begin{aligned}
O P^{2}=O Q^{2}-O G^{2}= & \left(\sigma_{1}^{2}+\sigma_{2}^{2}+\sigma_{3}\right)^{2}-\frac{1}{3}\left(\sigma_{1}+\sigma_{2}+\sigma_{3}\right)^{2} \\
& =\frac{1}{3}\left[\left(\sigma_{1}-\sigma_{2}\right)^{2}+\left(\sigma_{2}-\sigma_{3}\right)^{2}+\left(\sigma_{3}-\sigma_{1}\right)^{2}\right]
\end{aligned}
$$

For any given state of stress, the deviatoric stress vector will lie in the plane passing through $O$ and perpendicular to $O H$. This plane is known as the
deviatoric plane ( or sometimes $\pi$-plane )and its equation is $\sigma_{1}+\sigma_{2}+\sigma_{3}=0$ in the principal stress space. Since a uniform hydrostatic stress has no effect on yielding, it follows that yielding can depend only on the magnitude and direction of the deviatoric stress vector OP.

### 3.3 The Tresca and Mises criteria

Various criteria have been suggested in the past to predict the yielding of metals under complex stresses. Most of them are, however, only of historical interest, because they conflict with the experimental finding that a hydrostatic stress has no effect on yielding.
The two entirely satisfactory and widely used criteria are those due to Tresca and von Mises. From a series of experiments on the extrusion of metals, Tresca concluded that yielding occurred when the maximum shear stress reached a critical value; this value can be obtained from a simple experiment. For example, in a tension test, $\sigma_{1=} Y, \sigma_{2=} \sigma_{3=} 0$, where $Y$ is the yield stress in tension. In a shear test, $\sigma_{1=} T_{Y}, \sigma_{2=} 0, \sigma_{3=}-T_{Y}$ where $T_{Y}$ is the yield stress of a material in pure shear. The Tresca yield criterion may be written as:

$$
\sigma_{1}-\sigma_{3}=Y=2 \tau_{Y} \quad, \sigma_{1}>\sigma_{2}>\sigma_{3} \ldots \ldots \ldots \text { (1) }
$$

Thus, according to Tresca the yield surface is therefore a regular hexagonal cylinder which is inscribed within a cylinder of radius $\sqrt{2 / 3} Y$.

Von Mises suggested, from purely theoretical considerations, that yielding occurs when the elastic energy of distortion reaches a certain value at the yield point. The yield criterion proposed by von Mises may be written as:

$$
\begin{equation*}
\left(\sigma_{1}-\sigma_{2}\right)^{2}+\left(\sigma_{2}-\sigma_{3}\right)^{2}+\left(\sigma_{3}-\sigma_{1}\right)^{2}=2 Y^{2}=6 \tau_{Y}^{2} \tag{2}
\end{equation*}
$$

The yield surface is therefore a right cylinder of radius, $O P=\sqrt{2 / 3} Y$ and whose generators are perpendicular to the deviatoric plane.

The intersection of both cylinders with the m-plane is shown in the figure together with the projection of the three axes of principal stresses on that plane. It is customary to make the two criteria agree with each other in
uniaxial tension or compression, so that the Mises circle passes through the corners of the Tresca hexagon. The two yield loci differ most in a state of pure shear. For most metals, the yield criterion of von Mises defines the yield limit more accurately than does that of Tresca.


### 3.4 The plane stress yield locus

In a number of important physical problems, one of the principal stresses may be assumed to vanish. The yield criterion may then be represented by a closed curve where the nonzero principal stresses are plotted as rectangular coordinates. According to Tresca's yield criterion, the magnitude of the numerically greater of the two principal stresses is equal to $Y$ when these stresses are of the same sign, while the principal stress difference is of magnitude $Y$ when the stresses have opposite signs. Assuming $\sigma_{3}=0$, the Tresca yield locus in the ( $\sigma_{1}, \sigma_{2}$ ) plane is represented by a hexagon defined by the straight lines:
$\sigma_{1}= \pm Y \quad ; \quad \sigma_{2}= \pm Y \quad ; \quad \sigma_{1}-\sigma_{2}= \pm Y$

When $\sigma_{3}=0$, the von Mises yield criterion (2), expressed in terms of the principal stresses, reduces to:

$$
\sigma_{1}^{2}-\sigma_{1} \sigma_{2}+\sigma_{2}^{2}=Y^{2}
$$

which is the equation to an ellipse whose major and minor axes are inclined at an angle of $45^{0}$ with the $\sigma_{1}$ and $\sigma_{2}$
 axes (Figure). The Mises ellipse circumscribes the Tresca hexagon for a given uniaxial yield stress $Y$.

### 3.5 Experiments of Taylor and Quinney

In order to test whether the Von Mises or Tresca criteria best modelled the real behavior of metals, G I Taylor \& Quinney (1931), in a series of classic experiments, subjected a number of thin-walled cylinders made of copper and steel to combined tension and torsion, Figure.


The cylinder wall is in a state of plane stress, with $\sigma_{x x}=\sigma, \tau_{x y}=\tau$ and all other stress components zero. The principal stresses corresponding to such a stress-state are:

$$
\begin{aligned}
& \sigma_{1}=\frac{1}{2} \sigma+\frac{1}{2} \sqrt{\sigma^{2}+4 \tau^{2}} \\
& \sigma_{2}=\frac{1}{2} \sigma-\frac{1}{2} \sqrt{\sigma^{2}+4 \tau^{2}}
\end{aligned}
$$

and so Tresca's condition reduces to:

$$
\left(\frac{\sigma}{Y}\right)^{2}+\left(\frac{\tau}{Y / 2}\right)^{2}=1
$$

The
Mises
condition
reduces
to:

$$
\left(\frac{\sigma}{Y}\right)^{2}+\left(\frac{\tau}{Y / \sqrt{3}}\right)^{2}=1
$$

Thus both models predict an elliptical yield locus in $(\sigma, \tau)$ stress space, but with different ratios of principal axes, Figure. The origin in the figure corresponds to an

unstressed state. The horizontal axes refer to uniaxial tension in the absence of shear, whereas the vertical axis refers to pure torsion in the absence of tension. When there is a combination of $\sigma$ and $\tau$, one is off-axes. If the combination remains "inside" the yield locus, the material remains elastic; if the combination is such that one reaches anywhere along the locus, then plasticity ensues.
Taylor and Quinney, by varying the amount of tension and torsion, found that their measurements were closer to the Mises ellipse than the Tresca locus, a result which has been repeatedly confirmed by other workers.

### 3.6 Problems

1. A material is to be loaded to a stress state:

$$
\left[\begin{array}{lll}
\sigma_{x x} & \tau_{x y} & \tau_{x z} \\
\tau_{y x} & \sigma_{y y} & \tau_{y z} \\
\tau_{z x} & \tau_{z y} & \sigma_{z z}
\end{array}\right]=\left[\begin{array}{ccc}
50 & -30 & 0 \\
-30 & 90 & 0 \\
0 & 0 & 0
\end{array}\right] \mathrm{MPa}
$$

What should be the minimum uniaxial yield stress of the material so that it does not fail, according to the
(a) Tresca criterian
(b) Von Mises criterion

What do the theories predict when the yield stress of the material is 80 MPa ?
2. Suppose that, in the Taylor and Quinney tension-torsion tests, one has $\sigma=$ $0.5 Y$ and $\tau=0.433 Y$. Plot this stress state in the 2D principal stress state. Keeping now the normal stress at $0.5 Y$, what value can the shear stress be increased to before the material yields, according to the von Mises criterion?
3. An isotropic material exhibiting no Bauschinger effect is found to yield under biaxial stresses of 100 and -200 MPa. Show that the plane stress yield locus must pass through the stress points (100, -200), (-100, 200), (300, $100),(300,200),(-300,-100),(-300,-200)$, as well as those obtained by
interchanging each pair of coordinates, whatever the form of the yield criterion. Find the ratio of the uniaxial yield stress predicted by the Tresca criterion to that by the Mises criterion.
4. A closed-ended thin-walled tube of thickness $t$ and mean radius $r$ is subjected to an axial tensile force $P$, which is less than the value $P_{0}$ necessary to cause yielding. If a gradually increasing internal pressure $p$ is now applied, what value of $p$ such that the tube will yield according to the Tresca criterion when: i. $P=0.4 P_{0}$ and ii. $P=0.6 P_{0}$
5. A thin-walled tube with closed ends is subjected to an internal pressure $p$ as well as a torque that produces a shear stress $r$. If $p_{0}$ is the pressure required to produce a hoop stress equal to $Y$, show that yielding occurs according to the Mises criterion when:

$$
\left(\frac{p}{p_{0}}\right)^{2}+4\left(\frac{\tau}{Y}\right)^{2}=\frac{4}{3}
$$

## Stress-Strain Relations

### 4.1 Introduction

Once yield occurs, a material will deform plastically. Predicting and modelling this plastic deformation is the topic of this section. For the most part, in this section, the material will be assumed to be perfectly plastic, that is, there is no work hardening.

### 4.2 Plastic Strain Increments

When examining the strains in a plastic material, it should be emphasized that one works with increments in strain rather than a total accumulated strain. One reason for this is that when a material is subjected to a certain stress state, the corresponding strain state could be one of many. Similarly, the strain state could correspond to many different stress states. Examples of this state of affairs are shown in Figure.

One cannot therefore make use of stressstrain relations in plastic regions (except in some special

 cases), since there is no unique relationship between the current stress and the current strain. However, one can relate the current stress to the current increment in strain, and these are the "stress-strain" laws which are used in plasticity theory. The total strain can be obtained by summing up, or integrating, the strain increments.

### 4.3 The Prandtl-Reuss Equations

An increment in strain $d \varepsilon$ can be decomposed into an elastic part $d \varepsilon^{e}$ and a plastic part $d \varepsilon^{p}$. If the material is isotropic, it is reasonable to suppose that the principal plastic strain increments are proportional to the principal deviatoric stressess:

