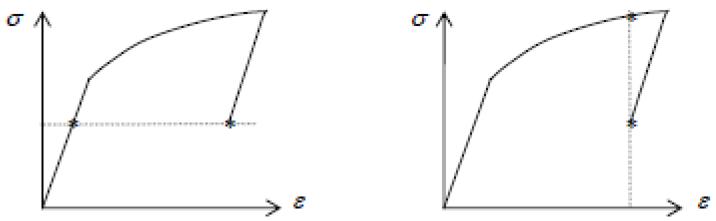
Stress-Strain Relations

4.1 Introduction

Once yield occurs, a material will deform plastically. Predicting and modelling this plastic deformation is the topic of this section. For the most part, in this section, the material will be assumed to be perfectly plastic, that is, there is no work hardening.

4.2 Plastic Strain Increments

When examining the strains in a plastic material, it should be emphasized that one works with *increments in strain* rather than a total accumulated strain. One reason for this is that when a material is subjected to a certain stress state, the corresponding strain state could be one of many. Similarly, the strain state could correspond to many different stress states. Examples of this state of affairs are shown in Figure.



One cannot therefore make use of stress-strain relations in plastic regions (except in some special cases), since there is no unique relationship between the current stress and the current strain. However, one can relate the current stress to the current **increment in strain**, and these are the "stress-strain" laws which are used in plasticity theory. The total strain can be obtained by summing up, or integrating, the strain increments.

4.3 The Prandtl-Reuss Equations

An increment in strain $d\varepsilon$ can be decomposed into an elastic part $d\varepsilon^e$ and a plastic part $d\varepsilon^p$. If the material is isotropic, it is reasonable to suppose that the principal plastic strain increments are proportional to the principal deviatoric stressess :

$$\frac{d\varepsilon_1{}^p}{\sigma_1'} = \frac{d\varepsilon_2{}^p}{\sigma_2'} = \frac{d\varepsilon_3{}^p}{\sigma_3'} = d\lambda \geq 0 \ldots\ldots\ldots (1)$$

This relation only gives the ratios of the plastic strain increments to the deviatoric stresses. To determine the precise relationship, one must specify the positive scalar *d*λ. Note that the plastic volume constancy is inherent in this relation:

$$d\varepsilon_1{}^p + d\varepsilon_2{}^p + d\varepsilon_3{}^p = 0$$

Eqns.(1) are in terms of the principal deviatoric stresses and principal plastic strain increments. In terms of Cartesian coordinates, one has:

$$\frac{d\varepsilon_{xx}{}^p}{s_{xx}} = \frac{d\varepsilon_{yy}{}^p}{s_{yy}} = \frac{d\varepsilon_{zz}{}^p}{s_{zz}} = \frac{d\gamma_{xy}{}^p}{\tau_{xy}} = \frac{d\gamma_{yz}{}^p}{\tau_{yz}} = \frac{d\gamma_{zx}{}^p}{\tau_{zx}} = d\lambda \geq 0 \ldots\ldots\ldots (2)$$

These equations are often expressed in the alternative forms:

$$\frac{d\varepsilon_{xx}{}^p - d\varepsilon_{yy}{}^p}{s_{xx} - s_{yy}} = \frac{d\varepsilon_{yy}{}^p - d\varepsilon_{zz}{}^p}{s_{yy} - s_{zz}} = \frac{d\varepsilon_{zz}{}^p - d\varepsilon_{xx}{}^p}{s_{zz} - s_{xx}} =$$

$$\frac{d\varepsilon_{xx}{}^p - d\varepsilon_{yy}{}^p}{\sigma_{xx} - \sigma_{yy}} = \frac{d\varepsilon_{yy}{}^p - d\varepsilon_{zz}{}^p}{\sigma_{yy} - \sigma_{zz}} = \frac{d\varepsilon_{zz}{}^p - d\varepsilon_{xx}{}^p}{\sigma_{zz} - \sigma_{xx}} = d\lambda \geq 0 \ldots\ldots\ldots (3)$$

In terms of actual stresses, one has, from (2):

$$d\varepsilon_{xx}{}^p = \frac{1}{3}\, d\lambda(2\sigma_{xx} - \sigma_{yy} - \sigma_{zz})$$

$$d\varepsilon_{yy}{}^p = \frac{1}{3}\, d\lambda(2\sigma_{yy} - \sigma_{zz} - \sigma_{xx})$$

$$d\varepsilon_{zz}{}^p = \frac{1}{3}\, d\lambda(2\sigma_{zz} - \sigma_{xx} - \sigma_{yy})$$

This *plastic* stress-strain law is known as a **flow rule**. Other flow rules are existing. The full elastic-plastic stress-strain relations are now, using Hooke's law:

$$d\varepsilon_{xx} = \frac{1}{E}(d\sigma_{xx} - \nu d\sigma_{yy} - \nu d\sigma_{zz}) + \frac{1}{3}\,d\lambda(2\sigma_{xx} - \sigma_{yy} - \sigma_{zz})$$

$$d\varepsilon_{yy} = \frac{1}{E}(d\sigma_{yy} - \nu d\sigma_{zz} - \nu d\sigma_{xx}) + \frac{1}{3}\,d\lambda(2\sigma_{yy} - \sigma_{zz} - \sigma_{xx})$$

$$d\varepsilon_{zz} = \frac{1}{E}(d\sigma_{zz} - \nu d\sigma_{xx} - \nu d\sigma_{yy}) + \frac{1}{3}\,d\lambda(2\sigma_{zz} - \sigma_{xx} - \sigma_{yy})$$

These expressions are called the **Prandtl-Reuss equations**. If the first, elastic, terms are neglected, they are known as the **Lévy-Mises** equations.
The magnitude of the plastic straining is determined by the multiplier $d\lambda$. This can be evaluated by noting that plastic deformation proceeds so long as the stress state remains on the yield surface, the so-called **consistency condition**. By definition, a perfectly plastic material is one whose yield surface remains unchanged during deformation.

4.4 The Yield Criterion Requirement

Note that one cannot propose a flow rule which gives the plastic strain increments as explicit functions of the stress, otherwise the yield criterion might not be met (in particular, when there is strain hardening); one must include the to-be-determined scalar plastic multiplier λ . The plastic multiplier is determined by ensuring the stress state lies on the yield surface during plastic flow.
When dealing with combined stress systems in the elastic-plastic range, it is common to use the *equivalent* or generalized stress-strain coordinates ($\bar{\sigma}, \bar{\varepsilon}$). Thus, the Tresca yield criterion may be written as:

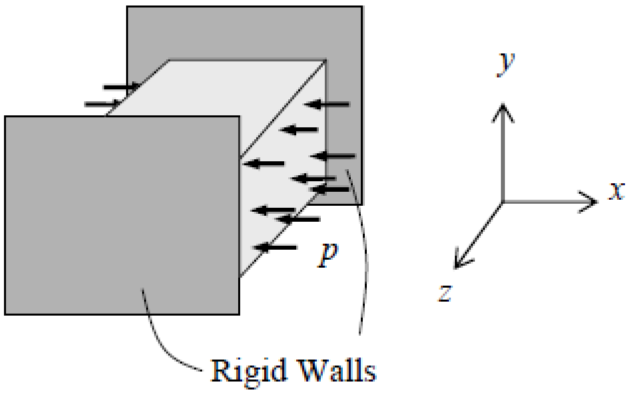
$$\bar{\sigma} = \sigma_1 - \sigma_3 = Y = 2\tau_Y \qquad , \quad \sigma_1 > \sigma_2 > \sigma_3$$

Similarly, Von Mises criterion may be written as:

$$\bar{\sigma} = \sqrt{\frac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]} = Y = \sqrt{3}\tau_Y$$

4.5 Example

Consider the constrained compression of a thick block, Figure. The block is subjected to an increasing pressure *p*.



The solution to the *elastic* problem is obtained as follows.

Since:

$\sigma_{xx} = -p$

$\sigma_{yy} = \varepsilon_{zz} = 0$

Thus:

$\sigma_{zz} = -\nu p$

$$\varepsilon_{xx} = -\frac{p}{E}(1 - \nu^2)$$

$$\varepsilon_{yy} = \frac{p}{E}\nu(1 + \nu)$$

and all other stress and strain components are zero. In this elastic phase, the principal stresses are clearly:

$$\sigma_1(= 0) > \sigma_2(= -\nu p) > \sigma_3(= -p)$$

The Prandtl-Reuss equations are, then:

$$d\varepsilon_{xx} = \frac{1}{E}(d\sigma_{xx} - \nu d\sigma_{zz}) + \frac{1}{3}\,d\lambda(2\sigma_{xx} - \sigma_{zz})$$

$$d\varepsilon_{yy} = \frac{1}{E}(-\nu d\sigma_{zz} - \nu d\sigma_{xx}) + \frac{1}{3}\,d\lambda(-\sigma_{zz} - \sigma_{xx})$$

$$d\varepsilon_{zz} = \frac{1}{E}(d\sigma_{zz} - \nu d\sigma_{xx}) + \frac{1}{3}\,d\lambda(2\sigma_{zz} - \sigma_{xx})$$

The Tresca yield criterion states that yield occurs when $\sigma_{xx} = -Y$, where Y is the uniaxial yield stress (in compression). Assume further perfect plasticity, so that $\sigma_{xx} = -Y$ holds during all subsequent plastic flow.

Thus, $d\sigma_{xx} = d\varepsilon_{zz} = 0$

The Prandtl-Reuss equations are now:

$$d\varepsilon_{xx} = \frac{1}{E}(-\nu d\sigma_{zz}) + \frac{1}{3}\,d\lambda(-2Y - \sigma_{zz})$$

$$d\varepsilon_{yy} = \frac{1}{E}(-\nu d\sigma_{zz}) + \frac{1}{3}\,d\lambda(-\sigma_{zz} + Y)$$

$$0 = \frac{1}{E}(d\sigma_{zz}) + \frac{1}{3}\,d\lambda(2\sigma_{zz} + Y)$$

Solving these equations, one can obtain:

$$d\lambda = -\frac{3}{E}\frac{d\sigma_{zz}}{2\sigma_{zz} + Y}$$

$$Ed\varepsilon_{xx} = -\nu d\sigma_{zz} + \frac{d\sigma_{zz}}{2\sigma_{zz}+Y}\,(2Y + \sigma_{zz})$$

$$Ed\varepsilon_{yy} = -\nu d\sigma_{zz} - \frac{d\sigma_{zz}}{2\sigma_{zz}+Y}\,(-\sigma_{zz} + Y)$$

Using the initial (yield point) conditions, i.e. $p = Y$ as follows:

$\sigma_{zz} = -\nu p = -\nu Y$

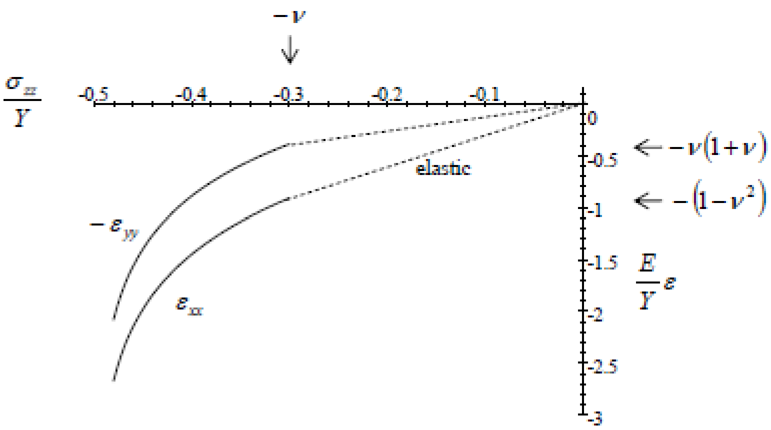
$$\varepsilon_{xx} = -\frac{p}{E}(1-\nu^2) = -\frac{Y}{E}(1-\nu^2)$$

$$\varepsilon_{yy} = \frac{p}{E}\nu(1+\nu) = \frac{Y}{E}\nu(1+\nu)$$

one can integrate to get:

$$\frac{E}{Y}\varepsilon_{xx} = -\frac{3}{4}ln\left(\frac{1-2\nu}{1+\frac{2\sigma_{zz}}{Y}}\right) + \frac{1}{2}(1-2\nu)\frac{\sigma_{zz}}{Y} - \frac{1}{2}(2-\nu) \qquad , \quad \frac{\sigma_{zz}}{Y} < -\nu$$

$$\frac{E}{Y}\varepsilon_{yy} = \frac{3}{4}ln\left(\frac{1-2\nu}{1+\frac{2\sigma_{zz}}{Y}}\right) + \frac{1}{2}(1-2\nu)\frac{\sigma_{zz}}{Y} + \frac{3}{2}\nu \qquad , \quad \frac{\sigma_{zz}}{Y} < -\nu$$



The stress-strain curves are shown in the Figure for v = 0.3. Note that, for a typical metal, $E\,/Y \sim 10^3$, and so the strains are very small right through the plastic compression; the plastic strains are of comparable size to the elastic strains. There is a rapid change of stress and then little change once σ_{zz} has approached close to its limiting value of $-\,Y\,/\,2$.

The above plastic analysis was based on σ_{xx} remaining the minimum principal stress.

This assumption has proved to be valid, since σ_{zz} remains between 0 and $-Y$ in the plastic region.

4.6 Problems

1. Consider the combined tension/torsion of a thin-walled tube as in the Taylor and Quinney tests. The tube is twisted up to the yield point. Torsion is then halted and tension is applied, holding the angle of twist constant. Take the Von Mises criterion and assume perfect plasticity, reduce the Prandtl-Reuss equations and give an expression for the normal strain.

2. Repeat the above question taking the Tresca criterion.

3. Consider the uniaxial straining of a perfectly plastic isotropic Von Mises metallic block. There is only one non-zero strain, ε_{xx}. One only need consider two stresses, σ_{xx}, σ_{yy} since $\sigma_{zz} = \sigma_{yy}$ by isotropy.

(i) Write down the two relevant Prandtl-Reuss equations.

(ii) Evaluate the stresses and strains at first yield.

(iii) For plastic flow, show that $d\sigma_{xx} = d\sigma_{yy}$ and that the plastic modulus is:

$$\frac{d\sigma_{xx}}{d\varepsilon_{xx}} = \frac{E}{3(1-2\nu)}$$

4. Consider the combined tension-torsion of a thin-walled cylindrical tube. The tube is made of a perfectly plastic Von Mises metal and Y is the uniaxial yield strength in tension. The axial strain is increased from zero until yielding occurs (with $\gamma_{xy} = 0$). From first yield, the axial strain is held constant and the shear strain is increased up to its final value of $(1+\nu)Y/\sqrt{3}E$

(i) Write down the yield criterion in terms of σ and τ only and sketch the yield locus in $\sigma - \tau$ space.

(ii) Evaluate the stresses and strains at first yield.

(iii)Evaluate $d\lambda$ in terms of $\sigma,\ d\sigma$.

(iv)Relate $\sigma,\ d\sigma$ to $\tau,\ d\tau$ and hence derive a differential equation for shear strain in terms of τ only

(v) Solve the differential equation and evaluate any constant of integration.

(vi)Evaluate the shear stress when γ_{xy} reaches its final value of $(1+\nu)Y/\sqrt{3}E$.

5. A closed-ended thin-walled tube of initial mean radius r_0 is subjected to an internal pressure *p*, and an external pressure *α p* on the cylindrical surface. The loading is continued into the plastic range by maintaining a constant value of *α* >0. Assuming the deformation to be uniform, and using the Lévy-Mises flow rate, determine the total equivalent strain $\bar{\varepsilon}$ at any stage as a function of *α* , *r* and r_0, where *r* is the current mean radius.

Chapter Five

Elastic-Plastic Bending of Beams

5.1 Introduction

In a deformable body subjected to external loads of gradually increasing magnitude, plastic flow begins at a stage when the yield criterion is first satisfied in the most critically stressed element. Further increase in loads causes spreading of the plastic zone which is separated from the elastic material by an elastic/plastic boundary. The position of this boundary is an unknown of the problem, and is generally so complicated in shape that the solution of the boundary-value problem often involves numerical methods.

When the design of components is based upon the elastic theory, e.g. the simple bending or torsion theory, the dimensions of the components are arranged so that the maximum stresses which are likely to occur under service loading conditions do not exceed the allowable working stress for the material in either tension or compression. The allowable working stress is taken to be the yield stress of the material divided by a convenient safety factor (usually based on design codes or past experience) to account for unexpected increase in the level of service loads. If the maximum stress in the component is likely to exceed the allowable working stress, the component is considered unsafe, yet it is evident that complete failure of the component is unlikely to occur even if the yield stress is reached at the outer fibres provided that some portion of the component remains elastic and capable of carrying load, i.e. the strength of a component will normally be much greater than that assumed on the basis of initial yielding at any position. To take advantage of the inherent additional strength, therefore, a different design procedure is used which is often referred to as ***plastic limit design.***