## Chapter Five

## Elastic-Plastic Bending of Beams

### 5.1 Introduction

In a deformable body subjected to external loads of gradually increasing magnitude, plastic flow begins at a stage when the yield criterion is first satisfied in the most critically stressed element. Further increase in loads causes spreading of the plastic zone which is separated from the elastic material by an elastic/plastic boundary. The position of this boundary is an unknown of the problem, and is generally so complicated in shape that the solution of the boundary-value problem often involves numerical methods.

When the design of components is based upon the elastic theory, e.g. the simple bending or torsion theory, the dimensions of the components are arranged so that the maximum stresses which are likely to occur under service loading conditions do not exceed the allowable working stress for the material in either tension or compression. The allowable working stress is taken to be the yield stress of the material divided by a convenient safety factor (usually based on design codes or past experience) to account for unexpected increase in the level of service loads. If the maximum stress in the component is likely to exceed the allowable working stress, the component is considered unsafe, yet it is evident that complete failure of the component is unlikely to occur even if the yield stress is reached at the outer fibres provided that some portion of the component remains elastic and capable of carrying load, i.e. the strength of a component will normally be much greater than that assumed on the basis of initial yielding at any position. To take advantage of the inherent additional strength, therefore, a different design procedure is used which is often referred to as plastic limit design.

### 5.2 Plastic bending of rectangular beams

A rectangular beam loaded until the yield stress has just been reached in the

outer fibres, Figure. Assume the beam is of an elastic perfectly plastic material.

The beam is still completely elastic and the bending theory applies, i.e.:
$M=\frac{\sigma I}{y}$
Thus, the maximum elastic moment: $M_{E}=Y \frac{B D^{2}}{6}$

If loading is then increased, it is assumed that instead of the stress at the outside increasing still further, more and more of the section reaches the yield stress Y .

Partially plastic moment, $M_{P P}=Y \frac{B d^{2}}{6}+Y \frac{B\left(D^{2}-d^{2}\right)}{4}$

When loading has been continued until the stress distribution is as assumed, the beam with collapse, the moment required to produce this fully plastic state is:

Fully plastic moment, $M_{F P}=Y \frac{B D^{2}}{4}$

This is the moment therefore which produces a plastic hinge in a rectangular beam.

### 5.3 Shape Factor

The shape factor is defined as the ratio of the moments required to produce fully plastic and maximum elastic states:
shape factor $=\frac{M_{F P}}{M_{E}}$

It is a factor which gives a measure of the increase in strength or loadcarrying capacity which is available beyond the normal elastic design limits for various shapes of section, e.g. for the rectangular section above,
shape factor $=Y \frac{B D^{2}}{4} / Y \frac{B D^{2}}{6}=1.5$
Thus rectangular beams can carry $50 \%$ additional moment to that which is required to produce initial yielding at the edge of the beam section before a fully plastic hinge is formed.

### 5.4 Partially plastic bending of unsymmetrical sections

Consider the T-section beam shown in Figure. Whilst stresses remain within the elastic limit the position of the N.A. can be obtained in the
 usual way.

Application of the simple bending theory about the N.A. will then yield the value of $\boldsymbol{M}_{\boldsymbol{E}}$ as described previously.
Whatever the state of the section, be it elastic, partially plastic or fully plastic, equilibrium of forces must always be maintained, i.e. at any section the tensile
forces on one side of the N.A. must equal the compressive forces on the other side.
$\Sigma$ stress x area above N.A. $=\Sigma$ stress x area below N.A.

In the fully plastic condition, therefore, when the stress is equal throughout the section, the above equation reduces to
$\Sigma$ areas above N.A. $=\Sigma$ areas below N.A

For all unsymmetrical sections the N.A. will move from its normal position when the section is completely elastic as plastic penetration proceeds. In the ultimate stage when a plastic hinge has been formed the N.A. will be positioned such that eqn. (1) applies. In the partially plastic state, as shown in Figure, the N.A. position is again determined by applying equilibrium
 conditions to the forces above and below the N.A. The section is divided into convenient parts, each subjected to a force $=$ average stress x area, as indicated, then:

$$
F_{1}+F_{2}=F_{3}+F_{4}
$$

The sum of the moments of these forces about the N.A. then yields the value of the partially plastic moment $M_{P P}$.

### 5.5 Collapse loads in beams

Having determined the moment required to produce a plastic hinge for the shape of beam cross-section used in any design it is then necessary to decide from a knowledge of the support and loading conditions how many such hinges are required before complete collapse of the beam or structure takes place, and to calculate the corresponding load. Here it is necessary to consider a plastic hinge as a pin-joint and to decide how many pin-joints are
required to convert the structure into a "mechanism". If there are a number of points of "local" maximum B.M., i.e. peaks in the B.M. diagram, the first plastic hinge will form at the numerical maximum; if further plastic hinges are required these will occur successively at the next highest value of maximum B.M. etc. It is assumed that when a plastic hinge has developed at any crosssection the moment of resistance at that point remains constant until collapse of the whole structure takes place owing to the formation of the required number of further plastic hinges.

## (a) Cantilever

There will only be one point of maximum B.M. and plastic collapse will occur with one plastic hinge at this point (Figure).


$P L \theta=M_{P} \theta$

Collapse load, $P=\frac{M_{P}}{L}$

## (b) Simply supported beam

There will only be one point of maximum B.M. and plastic collapse will occur with one plastic hinge at this point (Figure).


$$
P L \theta=2 M_{P} \theta
$$

Collapse load, $P=\frac{2 M_{P}}{L}$
(c) Built-in beam

In this case there are three positions of local maximum B.M., two of them being at the supports, and three plastic hinges are required for collapse
 (Figure).

$$
P L \theta=4 M_{P} \theta
$$

Collapse load, $P=\frac{4 M_{P}}{L}$

## (d) Propped cantilever

$$
P a \theta=2 M_{P} \alpha+M_{P} \theta
$$

Since: $a \theta=b \alpha$

and: $a+b=L$

Thus, Collapse load, $P=M_{P}\left(\frac{1}{a}+\frac{2}{L-a}\right)$

Minimum collapse load can be determined through $d P / d a=0$, i.e. when:

$$
a=(\sqrt{2}-1) L
$$

### 5.5.1 Residual stresses

In bending applications, when beams may be subjected to moments producing partial plasticity, i.e. part of the beam section remains elastic whilst
the outer fibres yield, this permanent set associated with the yielded areas prevents those parts of the material which are elastically stressed from returning to their unstressed state when load is removed. Residual stresses are therefore produced. In order to determine the magnitude of these residual stresses it is normally assumed that the unloading process, from either partially plastic or fully plastic states, is completely elastic (see Figure). The unloading stress distribution is therefore linear and it can be subtracted graphically from the stress distribution in the plastic or partially plastic state to obtain the residual stresses.


Consider, therefore, the rectangular beam which has been loaded to its fully plastic condition as represented by the stress distribution rectangles oabc and odef. The bending stresses which are then superimposed during the unloading process are given by the line goh and are opposite to sign. Subtracting the two distributions produces the shaded areas which then indicate the residual stresses which remain after unloading the plastically deformed beam. It should be observed that the loading and unloading moments must be equal, i.e. the moment of the force due to the rectangular distribution oabc about the $\mathrm{N} . \mathrm{A}$. must equal the moment of the force due to the triangular distribution oag. Now:

$$
\begin{gathered}
\frac{a g}{2} \times A \times \frac{2 o a}{3}=Y A \times \frac{o a}{2} \\
a g=1.5 Y
\end{gathered}
$$

Thus the residual stresses at the outside surfaces of the beam $=0.5 \mathrm{Y}$. The maximum residual stresses occur at the N.A. and are equal to the yield stress.

In loading cases where only partial plastic bending has occurred in the
 beam prior to unloading the stress distributions obtained are shown in Figure. Again, the unloading process is assumed elastic and the line goh in this case is positioned such that the moments of the loading and unloading stress distributions are once more equal, i.e. the stress at the outside fibre ag is determined by considering the plastic moment $M$ pp applied to the beam assuming it to be elastic; thus:

$$
a g=\frac{M_{P P}}{I} \frac{D}{2}
$$

In this case the maximum residual stress may occur either at the outside or at the inner boundary of the yielded portion depending on the depth of plastic penetration. There is no residual stress at the centre of the beam.

### 5.6 Collapse loads in oval links

The Figure shows a link symmetrical about centerline $A B$. To find the collapse load due purely to bending , and neglecting shear forces, assume plastic hinges as shown in the Figure for half of the link.

$$
P \cdot \overline{B B^{\prime}}=4 M_{P} \theta
$$



$$
\begin{aligned}
\overline{B B^{\prime}} & =\overline{N B^{\prime}}-\overline{B N} \\
& =\overline{C^{\prime} B^{\prime}} \cos (\alpha-\theta)-\overline{B N}
\end{aligned}
$$

Since: $\overline{C^{\prime} B^{\prime}}=\overline{C B}$
and: $\cos \theta=1$

$$
\sin \theta=\theta
$$

Thus, Collapse load,

$$
P=\frac{4 M_{P}}{\overline{C N}}
$$

### 5.7 Collapse loads in rings

The Figure shows a circular ring carrying three loads acting radially outwards and spaced uniformly at an interval of $2 \pi / 3$ radians. To find the collapse load due purely to bending , and neglecting shear forces, assume plastic hinges as shown in the Figure
 for one third of the ring.
$P \cdot \overline{B B^{\prime}}=4 M_{P} \theta$

$$
\begin{aligned}
\overline{B B^{\prime}} & =\overline{N B^{\prime}}-\overline{B N} \\
& =\frac{\sin (60+\theta)}{\sin 60} R-R
\end{aligned}
$$

Since: $\cos \theta=1$


$$
\sin \theta=\theta
$$

Thus, Collapse load,

$$
P=\frac{4 \sqrt{3} M_{P}}{R}
$$

### 5.8 Collapse loads in stud oval links

Assume a rigid stud is fixed diametrally as shown in the Figure. To find the collapse load due purely to bending, and neglecting shear forces, assume plastic hinges occur at $A, B$, and $C$ as shown in the Figure for one quarter of the link.


Following the principle of instantaneous center of rotation of the sub-link $B C$, one can get:
$P . \overline{I C} . \Omega=2 M_{P}(\omega+\Omega)+2 M_{P} \omega+2 M_{P} \Omega$

Since: $v_{B}=\overline{I B} . \Omega=\overline{A B} . \omega$

Thus, Collapse load,
$P=4 M_{P} \cdot \frac{\overline{A B}+\overline{I B}}{\overline{A B} \cdot \overline{I C}}=\frac{4 M_{P}}{\overline{B D}}$
$P$ is least when $\overline{B D}$ is greatest, which defines the position of hinge $B$. This is the point on the link at which the tangent is parallel to $\overline{A C}$.

### 5.9 Problems

1. Find the shape factor for a $150 \mathrm{~mm} \times 75$ mm channel in pure bending with the plane of bending perpendicular to the web of the
 channel. The dimensions are shown in Figure and $Z=21 \times 10^{-6} \mathrm{~m}^{3}$.
2. The cross-section of a beam is a channel, symmetrical about a vertical centre line. The overall width of the section is 150 mm and the overall depth 100 mm . The thickness of both the horizontal web and each of the vertical flanges is 12 mm . By comparing the behaviour in both the elastic and plastic range determine the shape factor of the section. Work from first principles in both cases.
3. The T -section beam shown in Figure is subjected to increased load so that yielding spreads to within 50 mm of the lower edge of the flange.

Determine the bending moment required to
 produce this condition.

$$
Y=240 \mathrm{MN} / \mathrm{m}^{2}
$$

4. (a) A rectangular section beam is 80 mm wide, 120 mm deep and is simply supported at each end over a span of 4 m . Determine the maximum uniformly distributed load that the beam can carry:
(i) if yielding of the beam material is permitted to a depth of 40 mm ;
(ii) before complete collapse occurs.
(b) What residual stresses would be present in the beam after unloading from condition (a) (i)?
The yield stress of the material of the beam $=280 \mathrm{MN} / \mathrm{m}^{2}$.
5. Determine the maximum intensity of loading that can be sustained by a simply supported beam, 75 mm wide $\times 100 \mathrm{~mm}$ deep, assuming elastic perfect -plastic behaviour with a yield stress in tension and compression of $135 \mathrm{MN} / \mathrm{m}^{2}$. The beam span is 2 m .
What will be the distribution of residual stresses in the beam after unloading?
6. A circular beam of length $L$, is cantilevered at both sides and loaded as shown.
i. Derive a formula for the load W
 required to set a complete collapse of the beam. ii. Determine the location of the load $W$ such that its value would be least.
7. An oval (elliptical) link, of rectangular cross section, is loaded and constrained as shown.
$a=2 b=100 \mathrm{~mm}, \mathrm{Mp}=100 \mathrm{Nm}$.
Determine the load $P$ required
 to set a complete collapse of the link.

## Elastic-Plastic Stresses in Thick Cylinder

### 6.1 Introduction

Thick cylinders are used as testing chambers or for the containment of fluid at high pressures. A number of important problems, such as the determination of stresses and strains in thick-walled pressure vessels are of this type. The axisymmetric problem is comparatively more difficult in principle, since there are three independent stress components, even when the stresses are assumed to vary only in the radial direction.

### 6.2 Lame equations

Consider the thick cylinder and the stresses acting on an element of unit length at radius $r$ are as shown in Figure.



For radial equilibrium of the element:

$$
\left(\sigma_{r}+d \sigma_{r}\right)(r+d r) d \theta-\sigma_{r} \times r d \theta=2 \sigma_{H} \times d r \times \sin \left(\frac{d \theta}{2}\right)
$$

