new value of the safety factor at the bore when the working pressure *P,* is applied.
The yield stress of the cylinder material Y = 850 MN/m^2 and axial stresses may be ignored.

2. A thick cylinder of inner radius of 60 mm and outer radius 190 mm, is constructed from material with a yield stress of 850 MN/m^2 and tensile strength 1 GN/m2. In order to prepare it for operation at a working pressure of 248 MN/m^2 it is subjected to an initial autofrettage pressure of 584 MN/m^2.
Ignoring axial stresses, compare the safety factors against initial yielding of the bore of the cylinder obtained with and without the autofrettage process.

3. A thick cylinder, of inner radius of 50 mm and outer radius 200 mm. What is the maximum autofrettage pressure which should be applied in order to achieve yielding to the geometric mean radius?
Determine the radius of zero hoop residual stress produced by the application and release of this pressure. Yield stress is 800 MN/m^2 .

4. A thick cylinder, of inner radius of 40 mm and outer radius 160 mm. What is the minimum autofrettage pressure which should be applied in order to achieve yielding through 90% of the cylinder thickness?
Determine the radius of maximum radial residual stress produced by the application and release of this pressure. Yield stress is 600 MN/m^2 .

Elastic-Plastic Stresses in Thick Sphere

7.1 Introduction

Thick spheres are used as testing chambers or for the containment of fluid at high pressures. A number of important problems, such as the determination of stresses and strains in thick-walled pressure vessels are of this type.

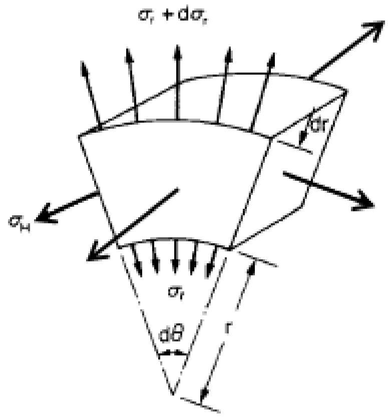
7.2 Lamé equations

Consider the thick sphere and the stresses acting on an element at radius *r* are as shown in Figure.

For radial equilibrium of the element:

$$(\sigma_r + d\sigma_r)((r+dr)d\theta)^2 - \sigma_r(rd\theta)^2 = 4\sigma_H . rdrd\theta \times \sin(\frac{d\theta}{2})$$

For small angles: $\sin(\frac{d\theta}{2}) \cong \frac{d\theta}{2}$



Therefore, neglecting second-order small quantities:

$$r\frac{d\sigma_r}{dr} = 2(\sigma_H - \sigma_r) \quad \dots\dots\dots(1)$$

$$\varepsilon_r = \frac{1}{E}(\sigma_r - \nu\sigma_H - \nu\sigma_H) = \frac{du}{dr} \dots\dots\dots(2)$$

$$\varepsilon_H = \frac{1}{E}(\sigma_H - \nu\sigma_r - \nu\sigma_H) = \frac{u}{r} \dots\dots\dots(3)$$

The three equations above, are solved and hence:

$$\frac{d}{dr}(2\sigma_H + \sigma_r) = 0$$

$$2\sigma_H + \sigma_r = constant = 3A \quad \dots\dots\dots(4)$$

Eqns.(1 and 4) yield:

$$r\frac{d\sigma_r}{dr} = 3A - 3\sigma_r$$

Therefore, integrating:

$3r^3(\sigma_r - A) = constant = 3B \ldots\ldots\ldots(5)$

Eqns.(4 and 5) yield: $\sigma_r = A + \frac{B}{r^3}$; $\sigma_H = A - \frac{B}{2r^3}$ (Lame' equations)

7.3 Internal pressure

Consider a thick sphere, internal radius *a* and external radius *b* , is subjected to an internal pressure *p* , the external pressure being zero.

At $r = a$, $\sigma_r = -p$ and at $r = b$, $\sigma_r = 0$

Substituting the above conditions in Lame' equations, one can get:

$$\sigma_H = p\,\frac{a^3(b^3+2r^3)}{2r^3(b^3-a^3)} \quad \ldots\ldots\ldots(6.1)$$

$$\sigma_r = -p\,\frac{a^3(b^3-r^3)}{r^3(b^3-a^3)} \quad \ldots\ldots\ldots(6.2)$$

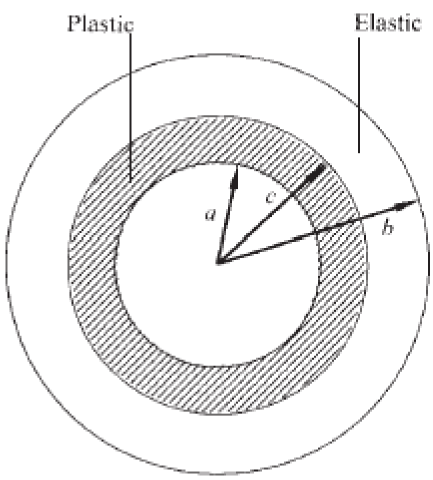
Now for ductile materials yield, using Tresca criterion, is deemed to occur when:

$$\sigma_H - \sigma_r = Y$$

The internal pressure required to cause yielding at the inner surface, *r* = *a* , is thus:

$$p = \frac{2Y}{3}\frac{b^3-a^3}{b^3}$$

The internal pressure required to cause yielding at an intermediate surface (Figure), *r* = *c* , is determined as follows:



At $r = c^-$, i.e. in the plastic area, the radial stress is determined by eqn.(1):

$$r\frac{d\sigma_r}{dr} = 2(\sigma_H - \sigma_r) = 2Y$$

Integrating with $\sigma_r = -p$ at $r = a$, the radial stress at any radius in the range $a \le r \le c$, is:

$$\sigma_r = 2Yln\frac{r}{a} - p$$

and at $r = c^-$, is:

$$\sigma_r = 2Yln\frac{c}{a} - p \dots\dots\dots (7)$$

In the elastic area, i.e. in the range $\le r \le b$, replace a by c and Lame' equations will be:

$$\sigma_H = p\,\frac{c^3(b^3+2r^3)}{2r^3(b^3-c^3)} \quad ; \quad \sigma_r = -p\,\frac{c^3(b^3-r^3)}{r^3(b^3-c^3)}$$

Tresca yield at $r = c^+$: $\quad \sigma_H - \sigma_r = p\,\frac{c^3(b^3+2r^3)}{2r^3(b^3-c^3)} + p\,\frac{c^3(b^3-r^3)}{r^3(b^3-c^3)} = Y$

or: $\quad p = \frac{2Y}{3}\frac{b^3-c^3}{b^3}$

This is the " internal" pressure required for the elastic portion of the cylinder in order to onset yielding at its inner surface, i.e. at $r = c^+$. In addition, this pressure is exerted by the radial stress at the outer surface of the plastic portion of cylinder, i.e. that given by eqn.(7). Thus:

$$2Yln\,\frac{c}{a} - p = -\,\frac{2Y}{3}\,\frac{b^3-c^3}{b^3}$$

or: $\quad p = 2Y\left(ln\,\frac{c}{a} + \frac{b^3-c^3}{3b^3}\right) \dots\dots\dots (8)$

which is the internal pressure in the cylinder required to bring yielding to a radius *c*.

In summary, following are the stresses induced in a thick sphere internally loaded with pressure of:

$$p = 2Y\left(ln\ \frac{c}{a} + \frac{b^3 - c^3}{3b^3}\right)$$

Plastic region: $a \leq r \leq c$

$$\sigma_H = 2Y\left(\frac{b^3 + 2c^3}{6b^3} - ln\ \frac{c}{r}\right) \quad \dots\dots\dots (9.1)$$

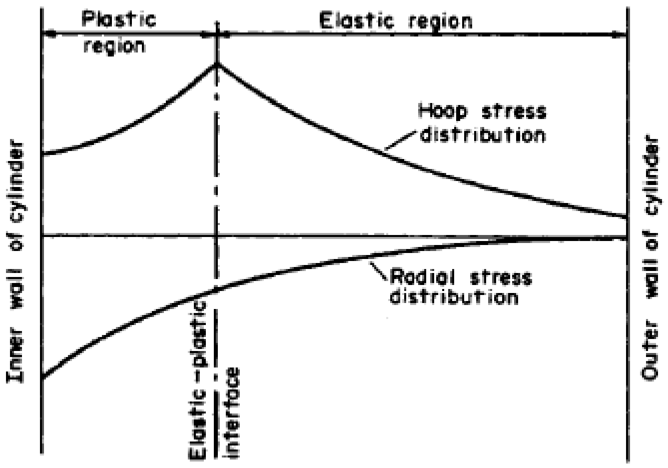
$$\sigma_r = -2Y\left(ln\ \frac{c}{r} + \frac{b^3 - c^3}{3b^3}\right) \dots\dots\dots (9.2)$$

Elastic region: $c \leq r \leq b$

$$\sigma_H = \frac{Y}{3}\frac{c^3(b^3 + 2r^3)}{b^3 r^3} \quad \dots\dots\dots (9.3)$$

$$\sigma_r = -\frac{2Y}{3}\frac{c^3(b^3 - r^3)}{b^3 r^3} \dots\dots\dots (9.4)$$

The above equations are shown in Figure.



7.4 Residual Stresses

If the pressure inside the sphere is increased beyond the initial yield value so that plastic penetration occurs only partly into the sphere wall then, on release

of the pressure, the elastic zone attempts to return to its original dimensions but is prevented from doing so by the permanent deformation or “set” of the yielded material. The result is that residual stresses are introduced, the elastic material being held in a state of residual tension whilst the inside layers are brought into residual compression. On subsequent loading cycles, therefore, the sphere is able to withstand a higher internal pressure since the compressive residual stress at the bore has to be overcome before this region begins to experience tensile stresses. This process is called ***autofrettage*** .
The autofrettage process has the same effect as shrinking one tube over another without the complications of the shrinking process. With careful selection of cylinder dimensions and autofrettage pressure the resulting residual compressive stresses can significantly reduce or even totally eliminate tensile stresses which would otherwise be achieved at the bore under working conditions. As a result the fatigue life and the safety factor at the bore are considerably enhanced and for this reason gun barrels and other pressure vessels are often pre-stressed in this way prior to service.
The autofrettage pressure required for yielding to any radius *c,* is given by eqn.(8) and if the unloading process is fully elastic, the unloaded elastic stresses are obtained
by applying a negative autofrettage pressure and substituting into eqns.(6), i.e.:

$$\sigma_H = -Y\left(ln\ \frac{c}{a} + \frac{b^3 - c^3}{3b^3}\right)\frac{a^3(b^3+2r^3)}{r^3(b^3-a^3)} \quad \ldots\ldots\ldots (10.1)$$

$$\sigma_r = 2Y\left(ln\ \frac{c}{a} + \frac{b^3 - c^3}{3b^3}\right)\frac{a^3(b^3-r^3)}{r^3(b^3-a^3)} \quad \ldots\ldots\ldots (10.2)$$

The residual stresses are, then, the sum of eqns. (9 and 10), i.e.:

$$p = 2Y\left(ln\ \frac{c}{a} + \frac{b^3 - c^3}{3b^3}\right)$$

Plastic region: $a \leq r \leq c$

$$\sigma_H = Y\left[\left(\frac{b^3+2c^3}{3b^3} - ln\ \frac{c}{r}\right) - \left(ln\ \frac{c}{a} + \frac{b^3-c^3}{3b^3}\right)\frac{a^3(b^3+2r^3)}{r^3(b^3-a^3)}\right]$$

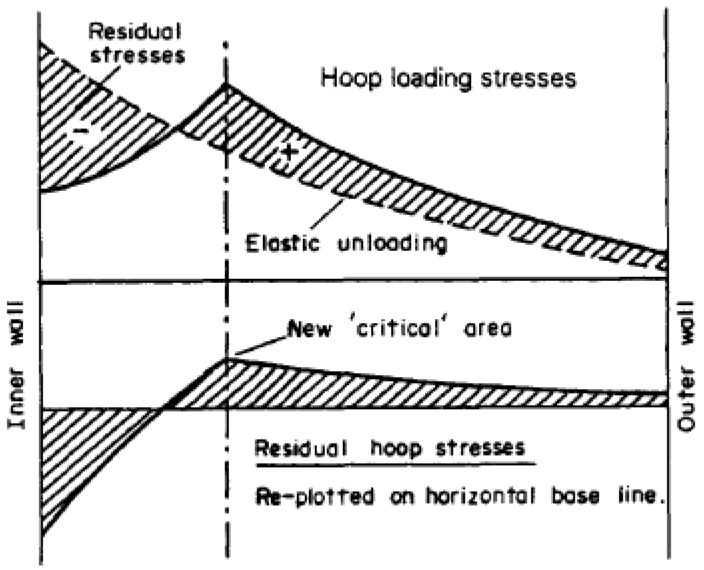
$$\sigma_r = -2Y\left[\left(ln\ \frac{c}{r} + \frac{b^3-c^3}{3b^3}\right) - \left(ln\ \frac{c}{a} + \frac{b^3-c^3}{3b^3}\right)\frac{a^3\left(b^3-r^3\right)}{r^3\left(b^3-a^3\right)}\right]$$

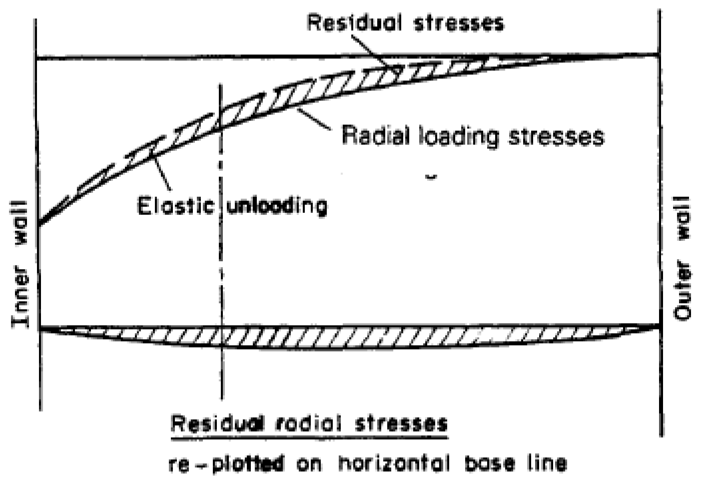
Elastic region: $c \leq r \leq b$

$$\sigma_H = Y\left[\frac{c^3(b^3+2r^3)}{3b^3r^3} - \left(ln\ \frac{c}{a} + \frac{b^3-c^3}{3b^3}\right)\frac{a^3(b^3+2r^3)}{r^3(b^3-a^3)}\right]$$

$$\sigma_r = -2Y\left[\frac{c^3\left(b^3-r^3\right)}{3b^3r^3} - \left(ln\ \frac{c}{a} + \frac{b^3-c^3}{3b^3}\right)\frac{a^3\left(b^3-r^3\right)}{r^3\left(b^3-a^3\right)}\right]$$

The above equations are shown in Figures.



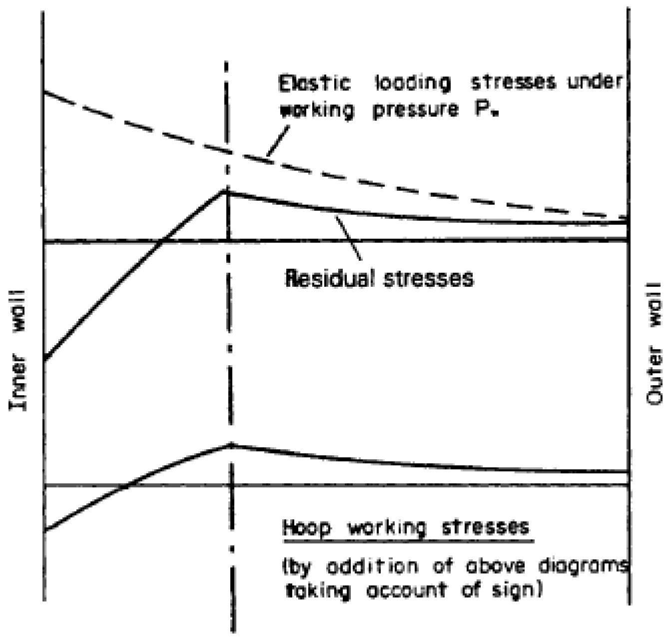


7.5 Working stress

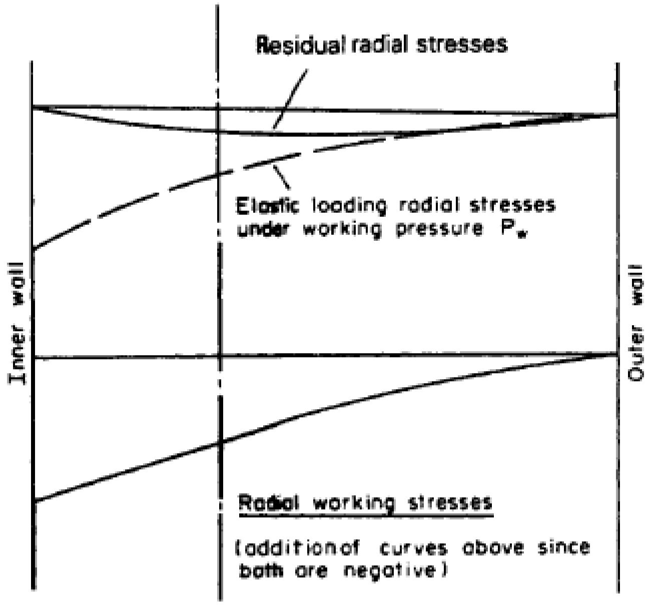
Finally, if the stresses due to an elastic internal working pressure P_W are superimposed on the residual stress state then the final working stress state is produced as in Figures.

The elastic working stresses are given by eqns.(3).

The final stress distributions



show that the maximum tensile stress, instead of being at the bore as in the plain cylinder, is now at the elastic/plastic interface position.



7.6 Problems

1. A thick sphere, inside radius 62.5 mm and outside radius 190 mm, forms the pressure vessel of an isostatic compacting press used in the manufacture of sparking plug components. Determine, using the Tresca theory of elastic failure, the safety factor on initial yield of the sphere when an internal working pressure P_W, of 240 MN/m^2 is applied.

(c) In view of the relatively low value of safety factor which is achieved at this working pressure, the sphere is now subjected to an autofrettage pressure of 580 MN/m^2. Determine the residual stresses produced at the inner surface of the sphere when the autofrettage pressure is removed and hence determine the new value of the safety factor at that surface when the working pressure P, is applied.

The yield stress of the sphere material Y = 850 MN/m^2 and axial stresses may be ignored.

2. A thick sphere of inner radius of 60 mm and outer radius 190 mm, is constructed from material with a yield stress of 850 MN/m^2 and tensile strength 1 GN/m2. In order to prepare it for operation at a working pressure of 248 MN/m^2 it is subjected to an initial autofrettage pressure of 584 MN/m^2.
Ignoring axial stresses, compare the safety factors against initial yielding of the inner surface of the sphere obtained with and without the autofrettage process.

3. A thick sphere, of inner radius of 50 mm and outer radius 200 mm. What is the maximum autofrettage pressure which should be applied in order to achieve yielding to the geometric mean radius?
Determine the radius of zero hoop residual stress produced by the application and release of this pressure. Yield stress is 800 MN/m^2 .

4. A thick sphere, of inner radius of 40 mm and outer radius 160 mm. What is the minimum autofrettage pressure which should be applied in order to achieve yielding through 90% of the sphere thickness?
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