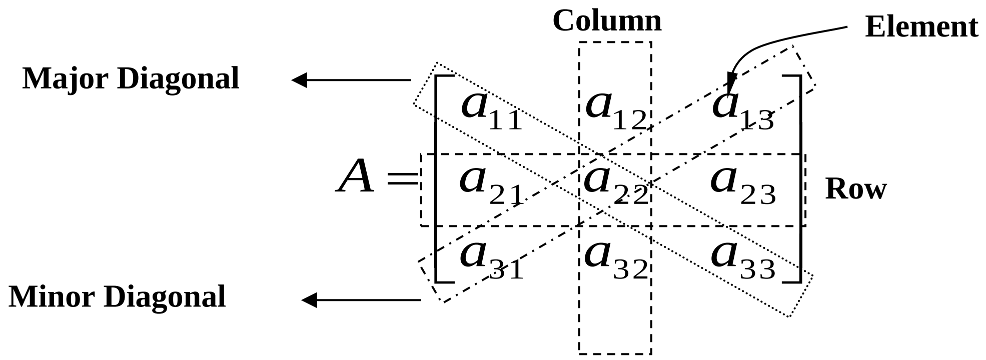
Ch.2: Solution Set of Simultaneous Equations

A- Direct Method
- **1-** Inverse of Matrix
- **2-** Gauss-Elimination Method
- **3-** Gauss-Jordan Method

B- Indirect Method
- **1-** Jacob's Method (Iterative Method)
- **2-** Gauss-Seidel Method

Introduction to Matrixes (You need to review Matrixes)



$$A=[a_{i,j}]$$ i=1, 2, 3, ……, n

J=1, 2, 3, ……., m

Matrixes Properties

1- Addition & Subtraction

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}; \quad B=\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

$$C=(A\pm B)=\begin{bmatrix} a_{11}\pm b_{11} & a_{12}\pm b_{12} & a_{13}\pm b_{13} \\ a_{21}\pm b_{21} & a_{22}\pm b_{22} & a_{23}\pm b_{23} \\ a_{31}\pm b_{31} & a_{32}\pm b_{32} & a_{33}\pm b_{33} \end{bmatrix}$$

2- Multiplication

In Multiplication process, the number of row in first matrix = the number of column in second matrix, as show following:

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}; B=\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}$$

$$C=(A*B)=\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}$$

$$c_{11} = (a_{11} * b_{11}) + (a_{12} * b_{21}) + (a_{13} * b_{31})$$
$$c_{12} = (a_{11} * b_{12}) + (a_{12} * b_{22}) + (a_{13} * b_{32})$$
$$c_{21} = (a_{21} * b_{11}) + (a_{22} * b_{21}) + (a_{23} * b_{31})$$
$$c_{22} = (a_{21} * b_{12}) + (a_{22} * b_{22}) + (a_{23} * b_{32})$$

Ex.1: Find the Multiplication of $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \end{bmatrix}$; $B = \begin{bmatrix} 3 & 2 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}$

Sol.

$$c_{11} = (1*3) + (2*2) + (1*3) = 10$$
$$c_{12} = (1*2) + (2*1) + (1*2) = 6$$
$$c_{21} = (2*3) + (3*2) + (2*3) = 18$$
$$c_{22} = (2*2) + (3*1) + (2*2) = 11$$

So, $C = (A * B) = \begin{bmatrix} 10 & 6 \\ 18 & 11 \end{bmatrix}$

3- Matrix Transpose

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}; \quad A^T \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

So, $A = [a_{i,j}] = A^T [a_{j,i}]$

4- Determinate

Let (A) matrix, Determinate of this matrix is (det A) or │A│ or Δ.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

│A│ = (a_{11}* a_{22}) − (a_{12}* a_{21})

Ex.2: Find the determinate of $A = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix}$.

Sol.

│A│ = (2*2) − (−3*3) = 13

Ex.3: Find the determinate of $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 1 \\ 3 & -1 & 2 \end{bmatrix}$

Sol.

First way $\Delta = 1\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix}$

Δ=(1)[(1*2) − (1*−1)] − (2) [(2*2)−(1*3)] + (−1)[(2*−1)− (1*3)]

Δ=(1)[(2) +(1)] − (2)[(4) +(3)] + (−1)[(−2) +(−1)]
Δ= 6

Second way In this case, we must add second & third Columns after third Column**.**

[∑ Multiplication of major diagonal − ∑ Multiplication of minor diagonal]

For following matrix:

$$\Delta = \begin{vmatrix} 1 & 2 & -1 & 1 & 2 \\ 2 & 1 & 1 & 2 & 1 \\ 3 & -1 & 2 & 3 & -1 \end{vmatrix} = [(1*1*2) + (2*1*3) + (-1*2*-1)] - [(3*1*-1) + (-1*1*1) + (2*2*2)] = 6$$

H.W. What's of meaning the following terms? (**You need to review Matrixes**)

Unit matrix, Zero matrix,
Null matrix, Transpose of matrix.
Upper Triangular matrix, Identity matrix,
Lower Triangular matrix, Submatrix for,
Diagonal matrix, Adjoint matrix,

Direct Method

1- Inverse of Matrix Method
2- Gauss-Elimination Method
3- Gauss-Jordan Method

1- Inverse of Matrix Method

Let $a_1 x + b_1 y + c_1 z = c_1$
$a_2 x + b_2 y + c_2 z = c_2$ → A.X=C → $X=A^{-1}.C$
$a_3 x + b_3 y + c_3 z = c_3$

Factors matrix $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ **Constants matrix**

Variables matrix

$[N]^T$ Transpose of Matrix Cofactor

$$A^{-1} = \frac{adj[A]}{|A|}, \quad adj[A] = [N]^T$$

and then " Check your answer"

Ex.4: Find the variables (x, y, z) by using **Matrix Inverse** for following of set linear Equs.

$$2x + y - z = 0$$
$$x - y + z = 6$$
$$x + 2y + z = 3$$

Sol. $\begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 2 & 1 \end{bmatrix} * \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 3 \end{bmatrix}$ → A.X=C → $X=A^{-1}.C$

$$\Delta = \begin{vmatrix} 2 & 1 & -1 & 2 & 1 \\ 1 & -1 & 1 & 1 & -1 \\ 1 & 2 & 1 & 1 & 2 \end{vmatrix} = [(2*-1*1)+(1*1*1)+(-1*1*2)]-[(1*-1*-1)+(2*1*2)+(1*1*1)] = -9$$

$$N = \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{bmatrix}$$

$$n_{11} = \begin{vmatrix} -1 & 1 \\ 2 & 1 \end{vmatrix} = -3 \qquad n_{12} = \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0 \qquad n_{13} = \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} = 3$$

$$n_{21}=\begin{vmatrix}1 & -1\\ 2 & 1\end{vmatrix}=3 \qquad n_{22}=\begin{vmatrix}2 & -1\\ 1 & 1\end{vmatrix}=3 \qquad n_{23}=\begin{vmatrix}2 & 1\\ 1 & 2\end{vmatrix}=3$$

$$n_{31}=\begin{vmatrix}1 & -1\\ -1 & 1\end{vmatrix}=0 \qquad n_{32}=\begin{vmatrix}2 & -1\\ 1 & 1\end{vmatrix}=3 \qquad n_{33}=\begin{vmatrix}2 & 1\\ 1 & -1\end{vmatrix}=-3$$

$$N=\begin{bmatrix}-3 & 0 & 3\\ 3 & 3 & 3\\ 0 & 3 & -3\end{bmatrix}\rightarrow\begin{bmatrix}+ & - & +\\ - & + & -\\ + & - & +\end{bmatrix}\rightarrow[N]^T\begin{bmatrix}-3 & 0 & 3\\ -3 & 3 & -3\\ 0 & -3 & -3\end{bmatrix}=adj[A]$$

$$A^{-1}=\frac{adj[A]}{|A|}=\frac{-1}{9}\begin{bmatrix}-3 & 0 & 3\\ -3 & 3 & -3\\ 0 & -3 & -3\end{bmatrix}$$

X=A^{-1}.C →

$$X=\frac{-1}{9}\begin{bmatrix}-3 & 0 & 3\\ -3 & 3 & -3\\ 0 & -3 & -3\end{bmatrix}*\begin{bmatrix}0\\ 6\\ 3\end{bmatrix}=\frac{-1}{9}\begin{bmatrix}-18\\ 9\\ -27\end{bmatrix}$$

So, the variables are; *x = 2, y = −1, z = 3*

Now, check your answer by substitution the values (x, y, z) in above Equs.

$$2x+y-z=2(2)+(-1)-(3)=0$$
$$x-y+z=(2)-(-1)+(3)=6$$
$$x+2y+z=(2)+2(-1)+(3)=3$$

Ex.5: Find the variables (x, y, z) by using **Matrix Inverse** for following of set linear Equs.

$$2x+2y=1$$
$$x+6y+z=0$$
$$y+z=1$$

Sol. $\begin{bmatrix}2 & 2 & 0\\ 1 & 6 & 1\\ 0 & 1 & 1\end{bmatrix}*\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$ → A.X=C → X=A^{-1}.C

$$\Delta = \begin{vmatrix} 2 & 2 & 0 & 2 & 2 \\ 1 & 6 & 1 & 1 & 6 \\ 0 & 1 & 1 & 0 & 1 \end{vmatrix} = [(2*6*1)+(2*1*0)+(0*1*1)]-[(0*6*0)+(1*1*2)+(1*1*2)] \rightarrow \Delta = 8$$

$$N = \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{bmatrix}$$

$$n_{11} = \begin{vmatrix} 6 & 1 \\ 1 & 1 \end{vmatrix} = 5 \qquad n_{12} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1 \qquad n_{13} = \begin{vmatrix} 1 & 6 \\ 0 & 1 \end{vmatrix} = 1$$

$$n_{21} = \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 2 \qquad n_{22} = \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} = 2 \qquad n_{23} = \begin{vmatrix} 2 & 2 \\ 0 & 1 \end{vmatrix} = 2$$

$$n_{31} = \begin{vmatrix} 2 & 0 \\ 6 & 1 \end{vmatrix} = 2 \qquad n_{32} = \begin{vmatrix} 2 & 0 \\ 1 & 1 \end{vmatrix} = 2 \qquad n_{33} = \begin{vmatrix} 2 & 2 \\ 1 & 6 \end{vmatrix} = 10$$

$$N = \begin{bmatrix} 5 & 1 & 1 \\ 2 & 2 & 2 \\ 2 & 2 & 10 \end{bmatrix} \rightarrow \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \rightarrow [N]^T = \begin{bmatrix} 5 & -1 & 1 \\ -2 & 2 & -2 \\ 2 & -2 & 10 \end{bmatrix} = adj[A]$$

$$A^{-1} = \frac{adj[A]}{|A|} = \frac{1}{8} \begin{bmatrix} 5 & -2 & 2 \\ -1 & 2 & -2 \\ 1 & -2 & 10 \end{bmatrix}$$

$$X = A^{-1}.C \quad \rightarrow \quad X = \frac{1}{8} \begin{bmatrix} 5 & -2 & 2 \\ -1 & 2 & -2 \\ 1 & -2 & 10 \end{bmatrix} * \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 7 \\ -3 \\ 11 \end{bmatrix}$$

So, the variables are; *x = 0.875, y = - 0.375, z = 1.375*

Now, check your answer by substitution the values (x, y, z) in above Equs.

$$2(0.875) + 2(-0.375) = 1$$
$$(0.875) + 6(-0.375) + (1.375) = 0$$
$$(-0.375) + (1.375) = 1$$

2- Gauss-Elimination Method

It's the basic of most other methods to solve set linear equations. In this method, we must obtain on "Upper Triangular matrix" from standard matrix by following steps:

1- Put the system of equations in matrix form.
$$\begin{vmatrix} a_{11} & a_{12} & a_{12} & d_{11} \\ a_{21} & a_{22} & a_{23} & d_{21} \\ a_{31} & a_{32} & a_{33} & d_{31} \end{vmatrix}$$

2- (a_{21}, a_{31}) must be equal to **zero** by following.

New Row (2) = Old Row (2) − Old Row (1)* $\frac{a_{21}}{a_{11}}$

New Row (3) = Old Row (3) − Old Row (1)* $\frac{a_{31}}{a_{11}}$

New Matrix $\begin{vmatrix} a_{11} & a_{12} & a_{12} & d_{11} \\ 0 & a_{22} & a_{23} & d_{21} \\ 0 & a_{32} & a_{33} & d_{31} \end{vmatrix}$

3- (a_{32}) must be equal to **zero** by following.

New Row (3) = Old Row (3) − Old Row (2)* $\frac{a_{32}}{a_{22}}$

New Matrix $\begin{vmatrix} a_{11} & a_{12} & a_{12} & d_{11} \\ 0 & a_{22} & a_{23} & d_{21} \\ 0 & 0 & a_{33} & d_{31} \end{vmatrix}$ this is Upper Triangular matrix

4- By Back Substitution to obtain the variables (x, y, z):

A_{33} (z) =d_{31} , a_{22} (y) +a_{23} (z)=d_{21} , a_{11} (x) +a_{12} (y)+ a_{13} (z)=d_{11}

and then" Check your answer"

<u>Ex.6:</u> Determine the variables (x, y, and z) by using **Gauss-Elimination Method** for following of set linear Equs.

$$3x - y + 2z = -3$$
$$x + y + z = -4$$
$$2x + y - z = -3$$

<u>Sol.</u> **1-** $\begin{vmatrix} a_{11} & a_{12} & a_{12} & d_{11} \\ a_{21} & a_{22} & a_{23} & d_{21} \\ a_{31} & a_{32} & a_{33} & d_{31} \end{vmatrix} \rightarrow \begin{vmatrix} 3 & -1 & 2 & -3 \\ 1 & 1 & 1 & -4 \\ 2 & 1 & -1 & -3 \end{vmatrix}$

2- New Row (2) = Old Row (2) − Old Row (1)* $\frac{a_{21}}{a_{11}}$ = 1− (3)* $\frac{1}{3} = 0$

= 1− (−1)* $\frac{1}{3} = 1.333$

$= 1 - (2) * \frac{1}{3} = 0.333$

$= -4 - (-3) * \frac{1}{3} = -3$

New Row (3) = Old Row (3) − Old Row (1)* $\frac{a_{31}}{a_{11}}$ $= 2 - (3) * \frac{2}{3} = 0$

$= 1 - (-1) * \frac{2}{3} = 1.666$

$= -1 - (2) * \frac{2}{3} = -2.333$

$= -3 - (-3) * \frac{2}{3} = -1$

New Matrix $\begin{vmatrix} 3 & -1 & 2 & -3 \\ 0 & 1.333 & 0.333 & -3 \\ 0 & 1.666 & -2.333 & -1 \end{vmatrix}$

3- New Row (3) = Old Row (3) − Old Row (2)* $\frac{a_{32}}{a_{22}}$

$= 1.666 - (1.333) * \frac{1.666}{1.333} = 0$

$= -2.333 - (0.333) * \frac{1.666}{1.333} = -2.75$

$= -1 - (-3) * \frac{1.666}{1.333} = 2.75$

New Matrix $\begin{vmatrix} 3 & -1 & 2 & -3 \\ 0 & 1.333 & 0.333 & -3 \\ 0 & 0 & -2.75 & 2.75 \end{vmatrix}$ this is Upper Triangular matrix

4- By Back Substitution to obtain the variables (x, y, z):

$-2.75z = 2.75 \rightarrow z = -1$
$1.333y - 0.333z = -3 \rightarrow y = -2$ "Check your answer"
$3x - y + 2z = -3 \rightarrow x = -1$

Ex.7: Using **Gauss-Elimination Method** to evaluate the variables (T_1, T_2, and T3) by for following of set linear Equs.

$$T_1 + T_2 + T_3 = 6$$
$$2T_1 - T_2 + T_3 = 3$$
$$-T_1 + 2T_2 - 2T_3 = -3$$

Sol. 1- $\begin{vmatrix} a_{11} & a_{12} & a_{12} & d_{11} \\ a_{21} & a_{22} & a_{23} & d_{21} \\ a_{31} & a_{32} & a_{33} & d_{31} \end{vmatrix} \rightarrow \begin{vmatrix} 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 3 \\ -1 & 2 & -2 & -3 \end{vmatrix}$

2- New Row (2) = Old Row (2) − Old Row (1)* $\frac{a_{21}}{a_{11}}$ = 2− (1)* $\frac{2}{1} = 0$

= −1− (−1)* $\frac{2}{1} = -3$

= 1− (1)* $\frac{2}{1} = -1$

= 3− (6)* $\frac{2}{1} = -9$

New Row (3) = Old Row (3) − Old Row (1)* $\frac{a_{31}}{a_{11}}$ = −1− (−1)* $\frac{-1}{1} = 0$

= 2− (1)* $\frac{-1}{1} = 3$

= − 2− (1)* $\frac{-1}{1} = -1$

= − 3− (6)* $\frac{-1}{1} = 3$

New Matrix $\begin{vmatrix} 1 & 1 & 1 & 6 \\ 0 & -3 & -1 & -9 \\ 0 & 3 & -1 & 3 \end{vmatrix}$

3- New Row (3) = Old Row (3) − Old Row (2)* $\frac{a_{32}}{a_{22}}$

=3− (−3)* $\frac{3}{-3} = 0$

= −1− (−1)* $\frac{3}{-3} = -2$

=3− (−9)* $\frac{3}{-3} = -6$

New Matrix $\begin{vmatrix} 1 & 1 & 1 & 6 \\ 0 & -3 & -1 & -9 \\ 0 & 0 & -2 & -6 \end{vmatrix}$ this is Upper Triangular matrix

4- By Back Substitution to obtain the variables (T_1, T_2, T_3):

$-2T_3 = -6 \rightarrow T_3 = 3$ " Check your answer"

$-3T_2 - T_3 = -9 \rightarrow T_2 = 2$

$T_1 + T_2 + T_3 = 6 \rightarrow T_1 = 1$

3- Gauss-Jordan Elimination Method

This method is similar to **Gauss- Elimination Method** but we must obtain a "Diagonal matrix" instead of "Upper Triangular matrix" by following steps:

1- Put the system of equations in matrix form.
$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & d_{11} \\ a_{21} & a_{22} & a_{23} & d_{21} \\ a_{31} & a_{32} & a_{33} & d_{31} \end{vmatrix}$$

2- (a_{21}, a_{31}) must be equal to **zero** by following.

New Row (2) = Old Row (2) − Old Row (1)* $\frac{a_{21}}{a_{11}}$

New Row (3) = Old Row (3) − Old Row (1)* $\frac{a_{31}}{a_{11}}$

New Matrix $\begin{vmatrix} a_{11} & a_{12} & a_{13} & d_{11} \\ 0 & a_{22} & a_{23} & d_{21} \\ 0 & a_{32} & a_{33} & d_{31} \end{vmatrix}$

3- (a_{32}) must be equal to **zero** by following:

New Row (3) = Old Row (3) − Old Row (2)* $\frac{a_{32}}{a_{22}}$

New Matrix $\begin{vmatrix} a_{11} & a_{12} & a_{13} & d_{11} \\ 0 & a_{22} & a_{23} & d_{21} \\ 0 & 0 & a_{33} & d_{31} \end{vmatrix}$ this is Upper Triangular matrix.

4- (a_{13}, a_{23}) must be equal to **zero** by following:

New Row (1) = Old Row (1) − Old Row (3)* $\frac{a_{13}}{a_{33}}$

New Row (2) = Old Row (2) − Old Row (3)* $\frac{a_{23}}{a_{33}}$

Also, new Matrix $\begin{vmatrix} a_{11} & a_{12} & 0 & d_{11} \\ 0 & a_{22} & 0 & d_{21} \\ 0 & 0 & a_{33} & d_{31} \end{vmatrix}$

5- (a_{12}) must be equal to **zero** by following:

New Row (1) = Old Row (1) − Old Row (2)* $\frac{a_{12}}{a_{22}}$

Now attain on Diagonal Matrix $\begin{vmatrix} a_{11} & 0 & 0 & d_{11} \\ 0 & a_{22} & 0 & d_{21} \\ 0 & 0 & a_{33} & d_{31} \end{vmatrix}$

6- By Back Substitution to obtain the variables (x, y, z):

a_{33}* z =d_{31} , a_{22}* y =d_{21} , a_{11}* x =d_{11} " Check your answer"

Ex.8: Using **Gauss- Jordan Elimination Method** to calculate the variables (P_1, P_2, and P_3) by for following of set linear Equs.

$$P_1 + P_2 + P_3 = 6$$
$$2P_1 - P_2 + P_3 = 3$$
$$-P_1 + 2P_2 - 2P_3 = -3$$

Sol. **1-** $\begin{vmatrix} a_{11} & a_{12} & a_{12} & d_{11} \\ a_{21} & a_{22} & a_{23} & d_{21} \\ a_{31} & a_{32} & a_{33} & d_{31} \end{vmatrix} \rightarrow \begin{vmatrix} 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 3 \\ -1 & 2 & -2 & -3 \end{vmatrix}$

2- New Row (2) = Old Row (2) − Old Row (1)* $\frac{a_{21}}{a_{11}}$ = 2− (1)* $\frac{2}{1} = 0$

= −1− (−1)* $\frac{2}{1} = -3$

= 1− (1)* $\frac{2}{1} = -1$

= 3− (6)* $\frac{2}{1} = -9$

New Row (3) = Old Row (3) − Old Row (1)* $\frac{a_{31}}{a_{11}}$ = −1− (−1)* $\frac{-1}{1} = 0$

= 2− (1)* $\frac{-1}{1} = 3$

= − 2− (1)* $\frac{-1}{1} = -1$

= − 3− (6)* $\frac{-1}{1} = 3$

New Matrix $\begin{vmatrix} 1 & 1 & 1 & 6 \\ 0 & -3 & -1 & -9 \\ 0 & 3 & -1 & 3 \end{vmatrix}$

3- New Row (3) = Old Row (3) − Old Row (2)* $\frac{a_{32}}{a_{22}}$ =3− (−3)* $\frac{3}{-3} = 0$

$= -1-(-1)*\dfrac{3}{-3} = -2$

$=3-(-9)*\dfrac{3}{-3} = -6$

New Matrix $\begin{vmatrix} 1 & 1 & 1 & 6 \\ 0 & -3 & -1 & -9 \\ 0 & 0 & -2 & -6 \end{vmatrix}$ this is Upper Triangular matrix

4- New Row (1) = Old Row (1) − Old Row (3)* $\dfrac{a_{13}}{a_{33}}$ $=1-(-2)*\dfrac{1}{-2} = 0$

$=6-(-9)*\dfrac{1}{-2} = 3$

New Row (2) = Old Row (2) − Old Row (3)* $\dfrac{a_{23}}{a_{33}}$ $=-1-(-2)*\dfrac{-1}{-2} = 0$

$=-6-(-9)*\dfrac{-1}{-2} = -6$

New Matrix $\begin{vmatrix} 1 & 1 & 0 & 3 \\ 0 & -3 & 0 & -6 \\ 0 & 0 & -2 & -6 \end{vmatrix}$

5- New Row (1) = Old Row (1) − Old Row (2)* $\dfrac{a_{12}}{a_{22}}$ $=1-(-3)*\dfrac{1}{-3} = 0$

$=3-(-6)*\dfrac{1}{-3} = 1$

Now attain on Diagonal Matrix $\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & -3 & 0 & -6 \\ 0 & 0 & -2 & -6 \end{vmatrix}$

6- By Back Substitution to obtain the variables (P_1, P_2, P_3):

$-2P_3 = -6$ → $P_3 = 3$

$-3P_2 = -6$ → $P_2 = 2$, $P_1 = 1$ " Check your answer"

Ex.9: Using **Gauss- Jordan Elimination Method** & **Gauss- Elimination Method** to calculate the variables (V_1, V_2, and V_3) by for following of set linear Equs.

$$4V_1 - 9V_2 + 2V_3 = 5$$
$$2V_1 - 4V_2 + 6V_3 = 3$$
$$V_1 - V_2 + 3V_3 = 4$$

Sol. 1- $\begin{vmatrix} a_{11} & a_{12} & a_{12} & d_{11} \\ a_{21} & a_{22} & a_{23} & d_{21} \\ a_{31} & a_{32} & a_{33} & d_{31} \end{vmatrix} \rightarrow \begin{vmatrix} 4 & -9 & 2 & 5 \\ 2 & -4 & 6 & 3 \\ 1 & -1 & 3 & 4 \end{vmatrix}$

2- New Row (2) = Old Row (2) − Old Row (1)* $\dfrac{a_{21}}{a_{11}}$ = 2− (4)* $\dfrac{2}{4}$ =0

= −4− (−9)* $\dfrac{2}{4}$ =0.5

= 6− (2)* $\dfrac{2}{4}$ =5

= 3− (5)* $\dfrac{2}{4}$ =0.5

New Row (3) = Old Row (3) − Old Row (1)* $\dfrac{a_{31}}{a_{11}}$ = 1− (4)* $\dfrac{1}{4}$ =0

= −1− (−9)* $\dfrac{1}{4}$ =1.25

= 3− (2)* $\dfrac{1}{4}$ =2.5

= 4− (5)* $\dfrac{1}{4}$ =2.75

New Matrix $\begin{vmatrix} 4 & -9 & 2 & 5 \\ 0 & 0.5 & 5 & 0.5 \\ 0 & 1.25 & 2.5 & 2.75 \end{vmatrix}$

3- New Row (3) = Old Row (3) − Old Row (2)* $\dfrac{a_{32}}{a_{22}}$ =1.25− (0.5)* $\dfrac{1.25}{0.5}$ =0

= 2.5− (5)* $\dfrac{1.25}{0.5}$ =−10

=2.75− (0.5)* $\dfrac{1.25}{0.5}$ =1.5

New Matrix $\begin{vmatrix} 4 & -9 & 2 & 5 \\ 0 & 0.5 & 5 & 0.5 \\ 0 & 0 & -10 & 1.5 \end{vmatrix}$ this is Upper Triangular matrix. By Back Substitution to obtain (V_1, V_2, and V_3) by **Gauss- Elimination Method**

$-10V_3=1.5 \rightarrow$ **$V_3=-0.15$**, $0.5V_2+5V_3=0.5\rightarrow$ **$V_2=2$**, $4V_1-9V_2+2V_3=5 \rightarrow$ **$V_1=6.95$**

4- New Row (1) = Old Row (1) − Old Row (3)* $\frac{a_{13}}{a_{33}}$ $=2-(-10)*\frac{2}{-10}=0$

$=5-(-1.5)*\frac{2}{-10}=5.3$

New Row (2) = Old Row (2) − Old Row (3)* $\frac{a_{23}}{a_{33}}$ $=5-(-10)*\frac{5}{-10}=0$

$=0.5-(1.5)*\frac{5}{-10}=1.25$

New Matrix $\begin{vmatrix} 4 & -9 & 0 & 5.3 \\ 0 & 0.5 & 0 & 1.25 \\ 0 & 0 & -10 & 1.5 \end{vmatrix}$

5- New Row (1) = Old Row (1) − Old Row (2)* $\frac{a_{12}}{a_{22}}$ $=-9-(0.5)*\frac{-9}{0.5}=0$

$=5.3-(1.25)*\frac{-9}{0.5}=27.8$

Now attain on Diagonal Matrix $\begin{vmatrix} 4 & 0 & 0 & 27.7 \\ 0 & 0.5 & 0 & 1.25 \\ 0 & 0 & -10 & 1.5 \end{vmatrix}$

6- By Back Substitution to obtain the variables (V_1, V_2, and V_3) by **Gauss- Jordan Elimination Method**:

$-10V_3 = 1.5 \quad \rightarrow V_3=-0.15$

$0.5V_2= 1.25 \quad \rightarrow V_2=2.5$

$-10V_1= 27.8 \quad \rightarrow V_1=6.95$ " Check your answer"

B- Indirect Method (Iterative Method)

1- Jacob's Method

2- Gauss-Seidel Method

1- Jacob's Method (Iterative Method)

It's the first ways to find a variables (x, y, z) because the easy using but slow reach to the right answer.

Let

$a_{11} x + a_{12} y + a_{13} z = c_{11}$

$a_{21} x + a_{23} y + a_{23} z = c_{21}$

$a_{31} x + a_{32} y + a_{33} z = c_{31}$

This equations system can be arrangement iteratively by following:

$$x_{n+1} = \frac{1}{a_{11}}[c_{11} - a_{12}.y_n - a_{13}.z_n]$$

$$y_{n+1} = \frac{1}{a_{23}}[c_{21} - a_{21}.x_n - a_{23}.z_n]$$; n=0, 1, 2, …. Old value

$$z_{n+1} = \frac{1}{a_{33}}[c_{31} - a_{31}.x_n - a_{32}.y_n]$$; n+1=1, 2, 3, …. New value

Solution steps

1- Calculate the first value of (x_1, y_1, z_1) by making initial value for (x_n, y_n, z_n)= (x_o, y_o, z_o)=(0, 0, 0).

2- The first value become old value for finding second value of (x_2, y_2, z_2) ..etc.

3- The solution is continued to reach to the right answer when $|x_{n+1} - x_n|$ & $|y_{n+1} - y_n|$ & $|z_{n+1} - z_n| \le \epsilon$.

4- To make the solution easy, make a table as following:

n	x_{n+1}	y_{n+1}	z_{n+1}
0			
1			
2			
.			
.			

"Check your answer"

<u>Ex.10:</u> Evaluate the variables (x, y, and z) by for following of set linear Equs. by using **Jacob's Method**

$$8x - y - z = 8$$

$$x - 7y + 2z_3 = -4$$

$$2x + y + 9z = 12$$

Sol. $(x_o, y_o, z_o)=(0, 0, 0)$

$$x_{n+1}=\frac{1}{a_{11}}[c_{11}-a_{12}.y_n-a_{13}.z_n] \rightarrow x_{n+1}=\frac{1}{8}[8+y_n+z_n]$$

$$y_{n+1}=\frac{1}{a_{23}}[c_{21}-a_{21}.x_n-a_{23}.z_n] \rightarrow y_{n+1}=\frac{1}{7}[4+x_n+2z_n]$$

$$z_{n+1}=\frac{1}{a_{33}}[c_{31}-a_{31}.x_n-a_{32}.y_n] \rightarrow z_{n+1}=\frac{1}{9}[12-2x_n-y_n]$$

n	x_{n+1}	y_{n+1}	z_{n+1}
0	1	0.571	1.333
1	1.238	1.095	1.047
2	1.267	1.047	0.936
3	1.247	1.012	0.935
4	1.243	1.016	0.943
5	1.244	1.018	0.944
6	1.245	1.018	0.943
7	1.245	1.018	0.943

x=1.245, y=1.018, z=0.943 " Check your answer"

Ex.11: Evaluate the variables (x, y, and z) by for following of set linear Equs. by using **Jacob's Method**

$$5T_1-2T_2+T_3=4$$
$$T_1+4T_2-2T_3=3$$
$$T_1+2T_2+4T_3=17$$

Sol. $(x_o, y_o, z_o)=(0, 0, 0)$

$$T^1_{n+1}=\frac{1}{a_{11}}[c_{11}-a_{12}.T^2_n-a_{13}.T^3_n] \rightarrow T^1_{n+1}=\frac{1}{5}[4+2T^2_n-T^3_n]$$, Let $x_{n+1}=T^1{}_{n+1}$, $y_{n+1}=T^2{}_{n+1}$, $z_{n+1}=T^3{}_{n+1}$

$$T^2_{n+1}=\frac{1}{a_{23}}[c_{21}-a_{21}.T^1_n-a_{23}.T^3_n] \rightarrow T^2_{n+1}=\frac{1}{4}[3-T^1_n+2T^3_n]$$

$$T^3_{n+1}=\frac{1}{a_{33}}[c_{31}-a_{31}.T^1_n-a_{32}.T^2_n] \rightarrow T^3_{n+1}=\frac{1}{4}[17-T^1_n-2T^2_n]$$

n	$T^1{}_{n+1}$	$T^2{}_{n+1}$	$T^3{}_{n+1}$
0	0.8	0.75	4.25
1	0.25	2.075	3.675
2	0.9	2.3375	3.15
3	1.105	1.429	2.856
4	0.8	1.073	3.261
5	0.577	1.58	3.315
6	0.73	1.93	3.315
?	1	2	3

T_1=1, T_2=2, T_3=3 " Check your answer"

2- Gauss-Seidel Method

It's the faster than **Jacob's Method** to find the variables, because the right side is contain on (x_{n+1}) in second, third equation as following:

Let

$a_{11}x + a_{12}y + a_{13}z = c_{11}$

$a_{21}x + a_{23}y + a_{23}z = c_{21}$

$a_{31}x + a_{32}y + a_{33}z = c_{31}$

$$x_{n+1} = \frac{1}{a_{11}}[c_{11} - a_{12}.y_n - a_{13}.z_n]$$

$$y_{n+1} = \frac{1}{a_{23}}[c_{21} - a_{21}.x_{n+1} - a_{23}.z_n] \quad ; n=0, 1, 2, \ldots \quad \text{Old value}$$

$$z_{n+1} = \frac{1}{a_{33}}[c_{31} - a_{31}.x_{n+1} - a_{32}.y_{n+1}] \quad ; n+1=1, 2, 3, \ldots \quad \text{New value}$$

Solution steps

1- Calculate the first value of (x_1, y_1, z_1) by making initial value for (x_n, y_n, z_n)= (x_o, y_o, z_o)=(0, 0, 0).

2- The first value become old value for finding second value of (x_2, y_2, z_2) ..etc.

3- The solution is continued to reach to the right answer when $|x_{n+1} - x_n|$ & $|y_{n+1} - y_n|$ & $|z_{n+1} - z_n| \le \epsilon$.

4- For solution easy, make a table as following:

n	x_{n+1}	y_{n+1}	z_{n+1}
0			
1			
2			
.			
.			

" Check your answer"

Ex.12: Find the variables (x, y, and z) by for following of set linear Equs. by using **Gauss-Seidel Method**

$$2x + 2y = 1$$
$$x + 6y + z = 0$$
$$y + z = 1$$

Sol. $$x_{n+1} = \frac{1}{a_{11}}[c_{11} - a_{12}.y_n - a_{13}.z_n] \rightarrow x_{n+1} = \frac{1}{2}[1 - 2y_n]$$

$$y_{n+1} = \frac{1}{a_{23}}[c_{21} - a_{21}.x_{n+1} - a_{23}.z_n] \rightarrow y_{n+1} = \frac{1}{6}[-2x_{n+1} - z_n]$$

$$z_{n+1} = \frac{1}{a_{33}}[c_{31} - a_{31}.x_{n+1} - a_{32}.y_{n+1}] \rightarrow z_{n+1} = \frac{1}{1}[1 - y_{n+1}]$$

n	x_{n+1}	y_{n+1}	z_{n+1}
0	0.5	- 0.166	1.166
1	0.666	-0.416	1.416
2	0.916	-0.541	1.541
3	1.041	-0.603	1.603
4	1.103	-0.634	1.634
5	1.134	-0.650	1.650

6	1.150	-0.658	1.658
7	1.158	-0.642	1.662
8	1.162	-0.664	1.664
9	1.164	-0.665	1.665
10	1.165	-0.665	1.665
11	1.165	-0.665	1.665

x=1.165, y=-0.665, z=1.665 " Check your answer"

Ex.13: Find the variables (x, y, and z) by for following of set linear Equs.
by using **Gauss-Seidel Method & W.H. Jacob's Method**

$$3x + y = 5$$

$$x + 2y = 5$$

Sol. $x_{n+1} = \frac{1}{a_{11}}[c_{11} - a_{12}.y_n - a_{13}.z_n] \rightarrow \quad x_{n+1} = \frac{1}{3}[5 - y_n]$

$$y_{n+1} = \frac{1}{a_{23}}[c_{21} - a_{21}.x_{n+1} - a_{23}.z_n] \rightarrow \quad y_{n+1} = \frac{1}{2}[5 - x_{n+1}]$$

n	x_{n+1}	y_{n+1}
0	1.666	1.667
1	1.111	1.944
2	1.018	1.990
3	1.003	1.998
4	1	2
5	1	2

x=1, y=2

"Check your answer"

Summary
Solution of Set Simultaneous Equ.

A-Direct Method
1-Inverse of Matrix Method

$$A^{-1} = \frac{adj[A]}{|A|}, \quad adj[A] = [N]^T$$

2-Gauss-Elimination Method

1- Put the system of equations in matrix form.

$$\begin{vmatrix} a_{11} & a_{12} & a_{12} & d_{11} \\ a_{21} & a_{22} & a_{23} & d_{21} \\ a_{31} & a_{32} & a_{33} & d_{31} \end{vmatrix}$$

2- (a_{21}, a_{31}) must be equal to **zero** by following.

New Row (2) = Old Row (2) − Old Row (1)* $\frac{a_{21}}{a_{11}}$

New Row (3) = Old Row (3) − Old Row (1)* $\frac{a_{31}}{a_{11}}$

New Matrix $\begin{vmatrix} a_{11} & a_{12} & a_{12} & d_{11} \\ 0 & a_{22} & a_{23} & d_{21} \\ 0 & a_{32} & a_{33} & d_{31} \end{vmatrix}$

3- (a_{32}) must be equal to **zero** by following.

New Row (3) = Old Row (3) − Old Row (2)* $\frac{a_{32}}{a_{22}}$

New Matrix $\begin{vmatrix} a_{11} & a_{12} & a_{12} & d_{11} \\ 0 & a_{22} & a_{23} & d_{21} \\ 0 & 0 & a_{33} & d_{31} \end{vmatrix}$ this is Upper Triangular matrix

4- By Back Substitution to obtain the variables (x, y, z):

a_{33}* z =d_{31} , a_{22}* y +a_{23}*z=d_{21} , a_{11}* x+a_{12}* y+a_{13}* z=d_{11}

3-Gauss-Jordan Elimination Method

It's similar to **Gauss-Elimination Method** additional to

4- (a_{12}) must be equal to **zero** by following:

New Row (1) = Old Row (1) − Old Row (2)* $\frac{a_{12}}{a_{22}}$

Now attain on Diagonal Matrix $\begin{vmatrix} a_{11} & 0 & 0 & d_{11} \\ 0 & a_{22} & 0 & d_{21} \\ 0 & 0 & a_{33} & d_{31} \end{vmatrix}$

5- By Back Substitution to obtain the variables (x, y, z):

a_{33}* z =d_{31} , a_{22}* y =d_{21} , a_{11}* x =d_{11}

B-Indirect Method

1- Jacob's Method (Iterative Method)

$$x_{n+1} = \frac{1}{a_{11}}[c_{11} - a_{12}.y_n - a_{13}.z_n]$$

$$y_{n+1} = \frac{1}{a_{23}}[c_{21} - a_{21}.x_n - a_{23}.z_n]$$

$$z_{n+1} = \frac{1}{a_{33}}[c_{31} - a_{31}.x_n - a_{32}.y_n]$$

n	x_{n+1}	y_{n+1}	z_{n+1}
0			
1			
2			
.			
.			

2- Gauss-Seidel Method

$$x_{n+1} = \frac{1}{a_{11}}[c_{11} - a_{12}.y_n - a_{13}.z_n]$$

$$y_{n+1} = \frac{1}{a_{23}}[c_{21} - a_{21}.x_{n+1} - a_{23}.z_n]$$

$$z_{n+1} = \frac{1}{a_{33}}[c_{31} - a_{31}.x_{n+1} - a_{32}.y_{n+1}]$$

n	x_{n+1}	y_{n+1}	z_{n+1}
0			
1			
2			
.			
.			

n=0, 1, 2, …. Old value ; n+1=1, 2, 3, …. New value