Ch.3: Numerical Integration

The area under curve is one of the common most application for "Numerical Integration" because the "Analytical Methods" are difficult solution or there's no solution in this way.

The most important Numerical Integration methods:

1- Trapezoidal Rule
2- Simpson Rule or Simpson's (1/3) Rule
3- Simpson's (3/8) Rule
4- Multiple integration by Trapezoidal, Simpson, and Simpson's (3/8) Rules

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1- Trapezoidal Rule

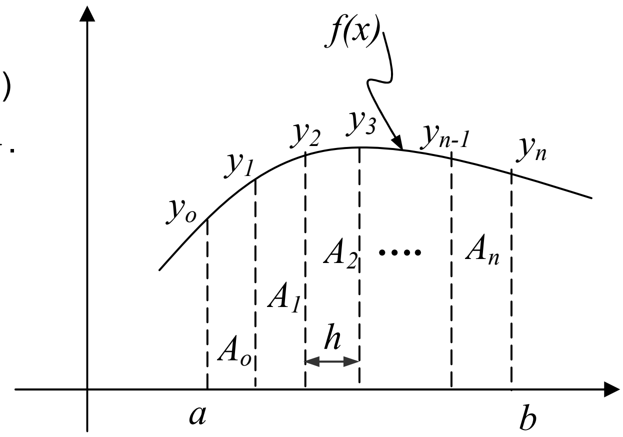
To find approximation value of a function *f(x)* for interval (*a, b*):

$A \; or \; I = \int_a^b f(x).dx$ by using Trapezoidal Rule:

1- Divided the range (*a, b*) to number of slips (*n*) which equal parts intervals it's have length $h = \frac{b-a}{n}$.

Now we can calculate trapezoidal area by:

$$A_o = \frac{1}{2}(y_1 + y_2).h$$



2- Find the total area between two intervals (*a, b*) by summation of all area from (x_o=*a*) to (x_n=*b*) by following:

$$A = I = \int_a^b f(x).dx = A_o + A_1 + A_2 + ... + A_n$$

$$= \frac{1}{2}h(y_o + y_1) + \frac{1}{2}h(y_1 + y_2) + \frac{1}{2}h(y_2 + y_3) + ... + \frac{1}{2}h(y_{n-1} + y_n)$$

$$A = I = \frac{1}{2}h[y_o + 2(y_1 + y_2 + y_3 + ... + y_{n-1}) + y_n]$$

3- The solution is good when $|I_{Numerical} - I_{Exact}| \le \epsilon$

Note: The accuracy of integration value is depended on the number of slips (*n*). When increase (*n*), increase accuracy of integration value.

Ex.1: Find the integral of function (x^2) by using "Trapezoidal Rule" when (n=4) for (0, 1) & satisfy this solution.

Sol.

By **Numerical** Integration:

$h = (b-a)/n = (1-0)/4 = 0.25$

$x_o=0 \quad \rightarrow y_o= f(0)=(0)^2=0$
$x_1=0.25 \quad \rightarrow y_1= f(0.25)= (0.25)^2=0.0625$
$x_2=0.5 \quad \rightarrow y_2= f(0.5)= (0.5)^2=0.\ 25$
$x_3=0.75 \quad \rightarrow y_3= f(0.75)= (0.75)^2=0.5625$
$x_4=1 \quad \rightarrow y_4= f(1)= (1)^2=1$

$$A\ or\ I=\int_a^b f(x).dx=\int_0^1 x^2.dx$$

$$A=\frac{1}{2}.h[y_o+2(y_1+y_2+y_3)+y_4]$$

$$=\frac{1}{2}*0.25[0+2(0.0625+0.25+0.5625)+1]$$

$$A=I=0.3437$$

By **Exact** solution: $A\ or\ I=\int_a^b f(x).dx=\int_0^1 x^2.dx=[\frac{x^3}{3}]_0^1=\frac{1}{3}=0.333$

$$|I_{Numerical}-I_{Exact}|=|0.3437-0.333|=0.0107=1.07\%$$

H.W. Solving the above example during (n=6, 9, 13) and compare with each integral result.

Ex.2: Determine the integral of function (*x.lnx*) by using "Trapezoidal Rule" while (*n*=6) for (0, 2) & satisfy this solution.

Sol.

By **Numerical** Integration:

$$h=(b-a)/n=(1-0)/6=0.333$$

x_o=0 → y_o= f(0)=0.ln(0)=0
x_1=0.333 → y_1= f(0.333)= 0.333 ln(0.333)= −0.3661
x_2=0.666 → y_2= f(0.666)= 0.666 ln(0.666)= −0.2707
x_3=0.999 → y_3= f(0.999)= 0.999 ln (0.999)= −0.001
x_4=1.332 → y_4= f(1.332)= (1.332) ln(1.332)= 0.3818
x_5=1.665 → y_5= f(1.665)= (1.665) ln(1.665)= 0.8488
x_6=2 → y_6= f(2)= (2) ln(2)= 1.3863

$$A\ or\ I=\int_a^b f(x).dx=\int_0^2 x.\ln(x).dx$$

$$A=\frac{1}{2}.h[y_o+2(y_1+y_2+y_3+y_4+y_5)+y_6]$$

$$=\frac{1}{2}*0.333[0+2(-0.3661-0.2707-0.001+0.3818+0.8488)+1.3863]$$

$$A=I=0.4282$$

By **Exact** solution:

$$A\ or\ I=\int_a^b f(x).dx=\int_0^2 x.\ln(x).dx$$ It's solving by Integral (udv)

$$\int u.dv=uv-\int v.du$$

$$Let\ \ u=\ln x$$

$$du=\frac{1}{x}dx$$

$$\int dv=x.dx$$

$$v=\frac{1}{2}x^2$$

$$\int_0^2 u.dv=\int_0^2 x.\ln(x).dx=\frac{1}{2}x^2.\ln x-\int\frac{1}{2}x^2.\frac{1}{x}dx$$

$$=[\frac{1}{2}x^2.\ln x-\frac{1}{4}x^2]_0^2$$

$$I=0.386$$

$$|I_{Numerical}-I_{Exact}|=|0.4282-0.386|=0.0422=4.22\%$$

Ex.3: A certain fluid has volume (0.08 m^3) is expansion reversibly in a cylinder behind a piston according to a law [*p.v=0.25*] to became (0.09 m^3), where (p) pressure in (kP) & (v) volume in (m^3). Find by Trapezoidal Rule work done from [$w=\int_{v1}^{v2} p\,dv$] & n=5.

Sol.

By **Numerical** Integration:

$$h=(b-a)/n=(0.09-0.08)/5=0.002$$

$v_o=0.08$ → $p_o= f(0.08)=0.25/0.08=3.125$
$v_1=0.082$ → $p_1= f(0.082)= 0.25/0.082=3.0487$
$v_2=0.084$ → $p_2= f(0.084)= 0.25/0.084=2.9762$
$v_3=0.086$ → $p_3= f(0.086)= 0.25/0.086=2.9070$
$v_4=0.088$ → $p_4= f(0.088)= 0.25/0.088=2.8410$
$v_5=0.09$ → $p_5= f(0.09) = 0.25/0.09 = 2.7777$

$$w\ or\ I=\int_a^b f(v).dx=\int_{0.08}^{0.09}\frac{0.25}{v}dv$$

$$A=\frac{1}{2}.h[p_o+2(p_1+p_2+p_3+p_4)+p_5]$$

$$=\frac{1}{2}*0.002[3.125+2(3.0487+2.9762+2.9070+2.8410)+2.7777]$$

$$w=0.02944\ kJ$$

By **Exact** solution: $w=\int_a^b f(v).dv=\int_{0.08}^{0.09}\frac{0.25}{v}\,dv=0.25[\ln v]_{0.08}^{0.09}=0.02944\ kJ$

$$|I_{Numerical}-I_{Exact}|=|0.02944-0.02944|=0\%$$

<u>Ex.4:</u> An 12m beam is subject to a load & shear force follows the equ. $V(x) = 5 + 0.25x^2$. Find the bending moment by $M = M_o + \int_0^x V.dx$, when M_o=0, n=6.

<u>Sol.</u>
By **Numerical** Integration:

$$h = (b-a)/n = (12-0)/6 = 2$$

$x_o=0$ → $V_o= f(0)=5+0.25\,(0)^2=5.25$
$x_1=2$ → $V_1= f(2)= 5+0.25\,(2)^2=6$
$x_2=4$ → $V_2= f(4)= 5+0.25\,(4)^2=9$
$x_3=6$ → $V_3= f(6)= 5+0.25\,(6)^2=14$
$x_4=8$ → $V_4= f(8)= 5+0.25\,(8)^2=21$
$x_5=10$ → $V_5= f(10) = 5+0.25\,(10)^2=30$
$x_6=12$ → $V_6= f(12) = 5+0.25\,(12)^2=36$

$$M = \int_a^b V.dx = \int_0^{12} [5 + 0.25x^2]\,dx$$

$$M = \frac{1}{2}.h[x_o + 2(x_1 + x_2 + x_3 + x_4 + x_5) + x_6]$$

$$= \frac{1}{2} * 2[5.25 + 2(6 + 9 + 14 + 21 + 30) + 36]$$

$$M = 201.25$$

<u>H.W.</u> Solving the following compare with each integral result by Trapezoidal Rule.

1- $x.e^{-x}$, (1, 2), n=4
2- $2-x^2$, (1, 2), n=6
3- $x/(1+x)$, (0, 1), n=2, 4
4- $1/(1+x^2)$ (0, 4), n=2, 6
5- $sin(x)$, (0^o, 90^o), n=8

By **Exact** solution: $M = \int_a^b V.dx = \int_{0.08}^{0.09} [5 + 0.25x^2]dx = [5x + \frac{0.25}{3}x^3]_0^{12} = 204$

$$|I_{Numerical} - I_{Exact}| = |204 - 201.25| = 0.0275\%$$

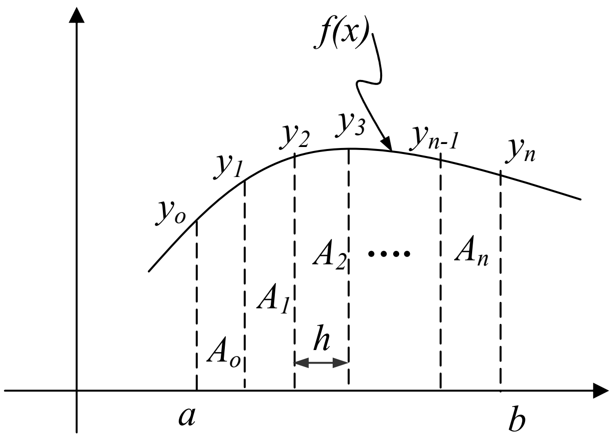
2- Simpson Rule or Simpson's (1/3) Rule

When the number of segments is **<u>even</u>**, we must use "Simpson's (1/3) Rule" to find integral of *f(x)* by following:

$$A\ or\ I=\int_a^b f(x).dx$$

$$A=\frac{1}{3}.h[y_o+4(y_1+y_3+...+y_{n-1})+2(y_2+y_4+....+y_{n-2})+y_n]$$

$$h=\frac{b-a}{n}$$



The solution steps of this method are similar to "Trapezoidal Rule".

<u>Ex.5:</u> Evaluate $\int_a^b e^{-x}.dx$ at range (0, 10) & No. parts (6) by Simpson Rule.

<u>Sol.</u> n=6 (Even) we must it's solving by **Simpson's (1/3) Rule.**

By **Numerical** Integration:

$$h=(b-a)/n=(10-0)/6=1.666$$

$x_o=0 \rightarrow y_o= f(0)=e^{-0} =1$
$x_1=1.666 \rightarrow y_1= f(1.666)= e^{-1.666}=0.1890$
$x_2=3.332 \rightarrow y_2= f(3.332)= e^{-3.332} =0.0356$
$x_3=4.998 \rightarrow y_3= f(4.998)= e^{-4.998}= 0.0067$
$x_4=6.664 \rightarrow y_4= f(6.664)= e^{-6.664}=0.00127$
$x_5=8.33 \rightarrow y_5= f(8.33)= e^{-8.33}= 0.00024$
$x_6=10 \rightarrow y_6= f(10)= e^{-10}=0.000045$

$$A\ or\ I=\int_a^b f(x).dx=\int_a^b e^{-x}.dx$$

$$=\frac{1}{3}.h[y_o+4(y_1+y_3+y_5)+2(y_2+y_4+y_6)+y_n]$$

$$=\frac{1}{3}.h[y_o+4(y_1+y_3+y_5)+2(y_2+y_4+y_6)+y_6]$$

$$=\frac{1}{3}*1.666[1+4(0.189+0.0067+0.00024)+2(0.0356+0.00127+0.000045)+0.000045]$$

$$I=1.031$$

By **Exact** solution: $I=\int_0^{10} e^{-x}.dx=-[e^{-x}]_0^{10}\rightarrow I=0.9999$

$|I_{Numerical} - I_{Exact}|=|1.031-0.9999|=0.0311=3.11\%$

Ex.6: Calculate the account of (x^4) as number of division (4) from initial value (0) to second initial integral (1) by Simpson Rule.

Sol. n=4 (Even) we must it's solving by **Simpson's (1/3) Rule.**

By **Numerical** Integration:

$$h = (b-a)/n = (1-0)/8 = 0.125$$

$x_o=0$ → $y_o= f(0)=(0)^4=0$
$x_1=0.125$ → $y_1= f(0.125)=(0.125)^4 =0.000244$
$x_2=0.25$ → $y_2= f(0.25)= (0.25)^4=0.004$
$x_3=0.375$ → $y_3= f(0.375)= (0.375)^4=0.02$
$x_4=0.5$ → $y_4= f(0.5)= (0.5)^4 =0.0625$
$x_5=0.625$ → $y_5= f(0.625)= (0.625)^4= 0.152$
$x_6=0.75$ → $y_6= f(0.75)= (0.75)^4=0.316$
$x_7=0.875$ → $y_7= f(0.875)= (0.875)^4=0.586$
$x_8=1$ → $y_8= f(1)= (1)^4=1$

$$A\ or\ I = \int_a^b f(x).dx = \int_a^b e^{-x}.dx$$

$$= \frac{1}{3}.h[y_o + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6 + y_8) + y_n]$$

$$= \frac{1}{3}.h[y_o + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6 + y_8) + y_8]$$

$$= \frac{0.125}{3}[0 + 4(0.00024 + 0.02 + 0.152 + 0.586) + 2(0.004 + 0.0625 + 0.316 + 1) + 1]$$

$$I = 0.283$$

By **Exact** solution: $I = \int_0^1 x^4.dx = \frac{1}{5}[x^5]_0^1 \rightarrow I = 0.2$

$|I_{Numerical} - I_{Exact}| = |0.283-0.2| = 0.083 = 8.3\%$

Ex.7: Evaluate the integral of following tabular data with 1- the Simpson Rule
2- the Trapezoidal Rule

x	-2	0	2	4	6	8	10
f(x)	35	5	-10	2	5	3	20

Sol. $h = (b-a)/n = [10-(-2)]/6 = 2$

$$A\ or\ I = \int_a^b f(x).dx = \int_{-2}^{10} f(x).dx$$

1- By Simpson Rule $= \frac{1}{3}.h[y_o + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4) + y_6]$

$$= \frac{2}{3}[35 + 4(5 + 2 + 3) + 2(-10 + 5) + 20] \rightarrow I = 56.6666$$

$$A \; or \; I = \int_a^b f(x).dx = \int_{-2}^{10} f(x).dx$$

2- Trapezoidal Rule $A = \frac{1}{2}.h[y_o + 2(y_1 + y_2 + y_3 + y_4 + y_5) + y_6]$

$$=[35 + 2(5 - 10 + 2 + 5 + 3) + 20]$$

$$A = I = 65$$

Ex.8: A certain fluid has volume (0.06 m^3) is expansion reversibly in a cylinder behind a piston according to a law [*p.v=0.5*] to became (0.09 m^3), where (*p*) pressure in (kP) & (*v*) volume in (m^3). Find by Simpson Rule work done from [$w = \int_{v1}^{v2} p\,dv$] & n=6.

Sol.

By **Numerical** Integration:

$$h = (b - a)/n = (0.09 - 0.06)/6 = 0.005$$

v_o=0.06 → p_o= f(0.06) =0.5/0.06 =8.3333
v_1=0.065 → p_1= f(0.065) = 0.5/0.065 =7.6923
v_2=0.07 → p_2= f(0.07) = 0.5/0.07 =7.1248
v_3=0.075 → p_3= f(0.075) = 0.5/0.075 =6.6666
v_4=0.08 → p_4= f(0.08) = 0.5/0.08 =6.25
v_5=0.085 → p_5= f(0.085) = 0.5/0.085 =5.8823
v_9=0.09 → p_5= f(0.09) = 0.5/0.09 =5.5555

$$w = \int_a^b p\,dv = \int_{0.06}^{0.09} \frac{0.5}{v}\,dx$$

$$= \frac{1}{3}.h[p_o + 4(p_1 + p_3 + p_5 + ...) + 2(p_2 + p_4 + p_6 + ...) + p_n]$$

$$= \frac{1}{3}.h[p_o + 4(p_1 + p_3 + p_5) + 2(p_2 + p_4) + p_6]$$

$$= \frac{0.005}{3}[8.3333 + 4(7.6923 + 6.6666 + 5.8823) + 2(7.1248 + 6.2500) + 5.5555]$$

$$w = 0.2026\,kJ$$

By **Exact** solution: $w = \int_a^b f(v).dv = \int_{0.06}^{0.09} \frac{0.5}{v}\,dv = 0.5[\ln v]_{0.06}^{0.09} = 0.2027\;kJ$

$\left| I_{Numerical} - I_{Exact} \right| = \left| 0.2026 - 0.2027 \right| = 0.0001 = 0.000001\%$

H.W. 1- Solving the above example during (n=2, 4, 6) and compare with each integral result by Simpson Rule.

2- x^2, (1, 3), n=4
3- *x.sin(x)*, (0, 2), n=6
4- $1/(1+ x^2)$, (0, 1), n=2, 4
5- *x/[sin(x)]*, (1, 5), n=2, 6

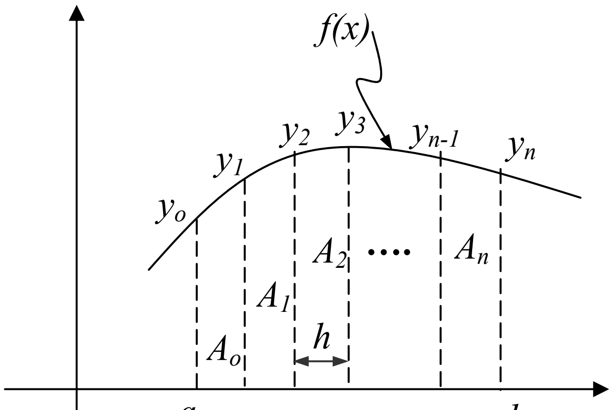
3- Simpson's (3/8) Rule

When the number of segments is **Odd**, the best solution used "Simpson's (3/8) Rule" to find integral of *f(x)* by following:

$$A\ or\ I = \int_a^b f(x).dx$$

$$A = \frac{3}{8}.h[y_o + 3(y_1 + y_2 + y_3 + y_4 + ...y_{n-1}) + y_n]$$

$$h = \frac{b-a}{n}$$



The solution steps of this method are also similar to "Trapezoidal Rule".

Ex.9: Calculate the account of (*1/x*) as number of sections (5) as the range beginning at (1) and finishing at (2), by Simpson Rule.

Sol. n=5 (Odd) the best solution used **"Simpson's (3/8) Rule"**.

By **Numerical** Integration:

$$h = (b-a)/n = (2-1)/5 = 0.2$$

$x_o=1$ → $y_o= f(1)=(1/1)=1$
$x_1=1.2$ → $y_1= f(1.2)=(1/1.2) =0.833$
$x_2=1.4$ → $y_2= f(1.4)= (1/1.4)=0.714$
$x_3=1.6$ → $y_3= f(1.6)= (1/1.6)=0.625$
$x_4=1.8$ → $y_4= f(1.8)= (1/1.8)=0.555$
$x_5=2$ → $y_5= f(2)= (1/2)= 0.5$

$$A\ or\ I = \int_a^b f(x).dx = \int_1^2 (1/x).dx$$

$$I = \frac{3}{8}.h[y_o + 3(y_1 + y_2 + y_3 + y_4 + ...y_{n-1}) + y_n]$$

$$= \frac{3}{8}.(0.2)[1 + 3(0.833 + 0.714 + 0.625 + 0.555) + 0.5]$$

$$I = 0.726$$

By **Exact** solution: $I = \int_1^2 (1/x)\,dx = [\ln x]_1^2 \rightarrow I = 0.693$

$|I_{Numerical} - I_{Exact}| = |0.726 - 0.693| = 0.033 = 3.3\%$

Ex.10: By Simpson Rule, calculate the amount of [*1/(1+x²)*] as number of sections (5) as the range prime at (0) and it's end at (1).

Sol. n=5 (Odd) the best solution used **"Simpson's (3/8) Rule"**.

By **Numerical** Integration:

$$h=(b-a)/n=(1-0)/5=0.2$$

$x_o=0$ → $y_o= f(0)=1$
$x_o=0.2$ → $y_1= f(0.2)=0.961$
$x_o=0.4$ → $y_2= f(0.4)=0.862$
$x_o=0.6$ → $y_3= f(0.6)=0.735$
$x_o=0.8$ → $y_4= f(0.8)=0.609$
$x_o=1$ → $y_5= f(1)= 0.5$

$$A\ or\ I=\int_a^b f(x).dx=\int_1^2 (1/x).dx$$

$$I=\frac{3}{8}.h[y_o+3(y_1+y_2+y_3+y_4+...y_{n-1})+y_n]$$

$$=\frac{3}{8}.(0.2)[1+3(0.961+0.862+0.735+0.609)+0.5]$$

$$I=0.825$$

By **Exact** solution: $I=\int_0^1 [1/(1+x^2)]dx=[\tan^{-1}(x)]_0^1 \rightarrow I=45^o=\frac{\pi}{4}=0.785$

$|I_{Numerical} - I_{Exact}| = |0.825 - 0.785| = 0.04 = 4.0\%$

H.W. 1- Solving the above example during (n=3, 7) and compare with each integral result by Simpson Rule.
2- x^2, (1, 3), n=5 3- *x/[sin(x)]*, (1, 5), n=3, 7
4- *x.sin(x)*, (0, 2), n=7 5- *1/(1+ x²)*, (0, 1), n=3, 5

2- A tank is discharging water through an orifice at a depth (*x*) meter below the surface of water whose area (A) m². The following are the values of the corresponding values of (A)

A	1.26	1.39	1.52	1.65	1.78	1.91	2.04
x	1.5	1.6	1.7	1.8	1.9	2	2.1

Using the formula $T=55.55\int\frac{A}{\sqrt{x}}dx$ to calculate (T) the time in second for the level of water to drop from (2.1m) to (1.5m) above the orifice. By 1- Simpson Rule n=5, 6 &
2-Trapezoidal Rule n=6, 7.

4- Multiple integration by Trapezoidal, and Simpson Rule

In this method, find the internal (inside) integral and then find the external integral or find the external integral and then find the internal (inside) integral by following:

$$A \ or\ I = \int_c^d\int_a^b f(x,y)\,dA = \int_c^d\int_a^b f(x,y)\,dx\,dy = \int_c^c\left[\int_a^b f(x,y)\,dx\right]dy = \int_a^b\left[\int_c^d f(x,y)\,dy\right]dx$$

$$x_o = a,\ x_n = b \quad ; \quad y_o = c,\ y_n = d \quad ; \quad h_x = \frac{b-a}{n_x} \quad ; \quad h_y = \frac{d-c}{n_y}$$

Better, we make a table to find x_o, x_1, x_2, ….x_n and y_o, y_1, y_2, …y_n.

Ex.11: By Trapezoidal Rule, calculate the double integral of [*x.y*] as number of sections (4) as the range prime at (0) and it's end at (2) for x-axis & as number of sections (4) as the range prime at (0) and it's end at (3) for y-axis.

Sol.

By **Numerical** Integration:

$$A \ or\ I = \int_c^d\int_a^b f(x,y)\,dA = \int_0^3\int_0^2 (x.y)\,dx\,dy$$

$$x_o = a = 0,\ x_n = b = 3 \quad ; \quad y_o = c = 0,\ y_n = d = 2 \quad ; \quad h_x = \frac{2-0}{4} = 0.5 \quad ; \quad h_y = \frac{3-0}{4} = 0.75$$

f(xy)

y_n \ x_n	x_o=0	x_1=0.5	x_2=1	x_3=1.5	x_4=2
y_o=0	0	0	0	0	0
y_1=0.75	0	0.375	0.75	1.125	1.5
y_2=1.5	0	0.75	1.5	2.25	3
y_3=2.25	0	1.125	2.25	3.375	4.5
y_4=3	0	1.5	3	4.5	6
I_x	0	1.5	3	4.5	6

$$I_x = \frac{1}{2}.h_x\,[y_o + 2(y_1 + y_2 + y_3) + y_4]$$

$$I_0 = 0$$

$$I_1 = \frac{1}{2} * 0.5\,[0 + 2(0.75 + 1.125 + 0.375) + 1.5] = 1.5$$

$$I_2 = \frac{1}{2} * 0.5\,[0 + 2(0.75 + 1.5 + 2.25) + 3] = 3$$

$$I_3 = \frac{1}{2} * 0.5\,[0 + 2(1.125 + 2.25 + 3.375) + 4.5] = 4.5$$

$$I_4 = \frac{1}{2} * 0.5\,[0 + 2(1.5 + 3 + 4.4) + 6] = 6$$

$$I = \frac{1}{2}.h_y\,[I_{x0} + 2(I_{x1} + I_{x2} + I_{x3}) + I_{x4}]$$

$$I = \frac{1}{2} * 0.75\,[0 + 2(1.5 + 3 + 4.5) + 6]$$

$$I = 9$$

By **Exact** solution:

$$A\ or\ I=\int_c^d\int_a^b f(x,y)\,dA=\int_0^3\int_0^2 (x.y)\,dx\,dy=\int_0^3\left[\frac{y}{2}.x^2\right]_0^2 dy=\int_0^3\left[\frac{y}{2}.(2)^2-\frac{y}{2}.(0)^2\right]dy=\int_0^3 2y\,dy$$

$$=\left[y^2\right]_0^3=[(3)^3-(0)^3]$$

$$I=9$$

So, Error=0

H.W. Solving the above example and compare with each integral result by Simpson Rule.

Ex.12: Determinate the double integral of [e^{y-x}] as number of parts (5) as the range prime at (0) and it's end at (1) for x-axis & as number of slips (5) as the range prime at (0) and it's end at (2) for y-axis. By Simpson Rule,

Sol.

By **Numerical** Integration: n=5 (Odd) the best solution used **"Simpson's (3/8) Rule"**.

$$A\ or\ I=\int_c^d\int_a^b f(x,y)\,dA=\int_1^2\int_0^1 (e^{y-x})\,dx\,dy \qquad h_x=\frac{1-0}{5}=0.2\ ;\quad h_y=\frac{2-1}{5}=0.2$$

f(e^{y-x})

y_n \ x_n	$x_o=0$	$x_1=0.2$	$x_2=0.4$	$x_3=0.6$	$x_4=0.8$	$x_5=1$
$y_o=1$	2.718	2.225	1.822	1.491	1.221	1
$y_1=1.2$	3.320	2.718	2.225	1.822	1.491	1.221
$y_2=1.4$	4.055	3.320	2.718	2.225	1.822	1.491
$y_3=1.6$	4.953	4.055	3.320	2.718	2.225	1.822
$y_4=1.8$	6.049	4.953	4.055	3.320	2.718	2.225
$y_5=2$	7.389	6.049	4.953	4.055	3.320	2.718
I_x	4.892	4.006	3.279	2.685	2.198	1.799

$$I_x=\frac{3}{8}.h_x[y_o+3(y_1+y_2+y_3+y_4)+y_5]$$

$$I_0=\frac{3}{8}*0.2[2.718+3(3.320+4.055+4.953+6.049)+7.389]=4.892$$

$$I_1=\frac{3}{8}*0.2[2.225+3(2.718+3.320+4.055+4.953)+6.049]=4.006$$

$$I_2=\frac{3}{8}*0.2[1.822+3(2.225+3.320+2.718+4.055)+4.953]=3.279$$

$$I_3=\frac{3}{8}*0.2[1.491+3(1.822+2.225+3.220+2.719)+4.055]=2.685$$

$$I_4=\frac{3}{8}*0.2[1.221+3(1.822+2.225+2.718+1.491)+3.320]=2.198$$

$$I_5=\frac{3}{8}*0.2[1+3(1.221+1.493+1.822+2.225)+2.718]=1.799$$

$$I = \frac{3}{8}.h_y\,[I_{x0} + 3(I_{x1} + I_{x2} + I_{x3} + I_{x4}) + I_{x5}]$$

$$I = \frac{3}{8} * 0.2[4.892 + 3(4.006 + 3.279 + 2.685 + 2.198) + 1.799]$$

$$I = 3.239$$

By **Exact** solution:

$$A\ or\ I = \int_c^d\int_a^b f(x,y)\,dA = \int_1^2\int_0^1 (e^{y-x})\,dx\,dy = \int_1^2 \left[-e^{y-x}\right]_0^1 dy = \int_1^2 \left[-e^{y-1} + e^{y}\right]dy = \left[-e^{y-1} + e^{y}\right]_1^2$$

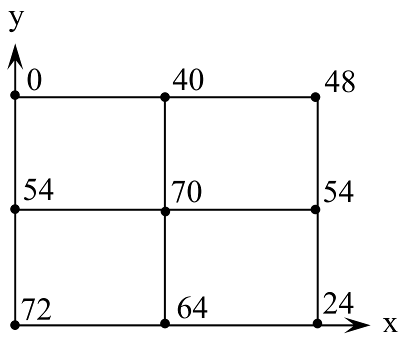
$$I = 2.962$$

So, Error=0.3=30%

<u>H.W.</u> Solving the above example and compare with each integral result by Trapezoidal Rule.

<u>Ex.12:</u> Suppose that the temperature of a rectangular heated plate is described by the following function: $T(x,y) = 2xy+2x-x^2-2y^2+72$. If plate is 8m long & 6m wide, compute the average temperature.

<u>Sol.</u>



Summary
Solution of Set Simultaneous Equ.

1- Trapezoidal Rule

$$A = I = \int_a^b f(x).dx = A_o + A_1 + A_2 + \ldots + A_n$$

$$= \frac{1}{2}h(y_o + y_1) + \frac{1}{2}h(y_1 + y_2) + \frac{1}{2}h(y_2 + y_3) + \ldots. + \frac{1}{2}h(y_{n-1} + y_n)$$

$$A = I = \frac{1}{2}h[y_o + 2(y_1 + y_2 + y_3 + \ldots. + y_{n-1}) + y_n]$$

2- Simpson Rule or Simpson's (1/3) Rule (<u>n=Even</u>)

$$A\ or\ I = \int_a^b f(x).dx$$

$$A = \frac{1}{3}.h[y_o + 4(y_1 + y_3 + \ldots) + 2(y_2 + y_4 + \ldots.) + y_n]$$

3- Simpson Rule or Simpson's (3/8) Rule (<u>n=Odd</u>)

$$A\ or\ I = \int_a^b f(x).dx$$

$$A = \frac{3}{8}.h[y_o + 3(y_1 + y_2 + y_3 + y_4 + \ldots y_{n-1}) + y_n]$$

4- When segments (h) <u>aren't equal</u> for all range of integral

Tare are two ways to solve:

a- By just Trapezoidal Rule

b- By Trapezoidal, Simpson's (1/3), Simpson's (3/8) Rules

5- Multiple integration by Trapezoidal, and Simpson

y_n \ x_n	x_o=	x_1=	x_n=
y_o=			
y_1=			
yn=			
I_x			

For all cases

$$x_o = a,\ x_n = b \quad ; \quad y_o = c,\ y_n = d \quad ; \quad h_x = \frac{b-a}{n_x} \quad ; \quad h_y = \frac{d-c}{n_y}$$

The solution is good when $|I_{Numerical} - I_{Exact}| \le \epsilon$

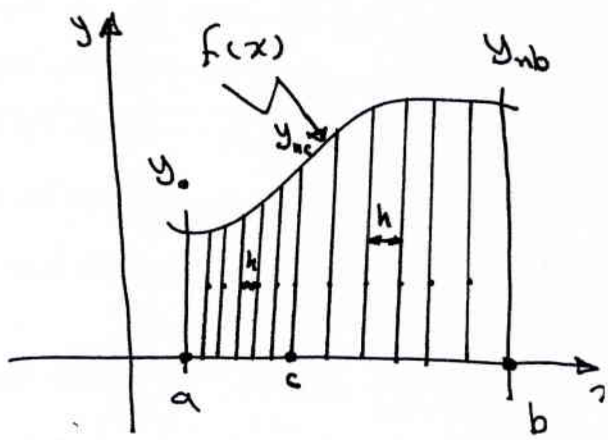
4. Integration with Unequal Segments (h) :-

The previous meths are solving the Numerical integral when the segment (h) is equal for all function domin (a,b) or (x_0, x_n), but when the Element (h) is Unequal; as in exprimental test; we must use [Trapizoidal Method] to find the area under curve by this equation.

$$I = A = h_1 \frac{y_0 + y_1}{2} + h_2 \frac{y_1 + y_2}{2} + h_3 \frac{y_2 + y_3}{2} + \cdots + h_n \frac{y_{n-1} + y_n}{2}$$

When the elements (h) are equal from $(a \to c)$ & the elements (h) are equal from $(c \to b)$; but, Both elements $(h)_{a \to c}$ & $(h)_{b \to c}$ are diffirent, we must care to "number of elements" & the width of element (h).

* When the number of elements (n) are even, we must use "Simpson's (1/3) Rule"
* When the number of elements (n) are odd, we must use "Simpson's (3/8) Rule"



$$I_3 = \frac{3}{8}h\left[y_0 + 3(y_1 + y_2) + y_4\right] = \frac{3}{8}(0.04)\left[0.43 + 3(.75 + .56) + 0.42\right]$$

$$I_3 = 0.0717$$

$$I_4 = \frac{1}{3}h\left[y_0 + 4y_2 + y_3\right] = \frac{1}{3}(0.1)\left[0.42 + 4(0.72) + 0.819\right] \rightarrow I_4 = 0.137$$

$$I_5 = h\,\frac{y_8 + y_9}{2} = 0.06\,\frac{0.819 + 0.63}{2} \rightarrow I_5 = 0.04347$$

$$I_6 = h\,\frac{y_9 + y_{10}}{2} = 0.1\,\frac{0.63 + 0.7}{2} \rightarrow I_6 = 0.0665$$

$$I = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 \rightarrow \boxed{I = 0.555}$$

Ex.11: Determine the integral of data in the following

x	0.0	0.12	0.22	0.32	0.36	0.4	0.44	0.54	0.64	0.7	0.8
$f(x)$	0.2	0.97	0.52	0.43	0.75	0.56	0.42	0.72	.819	.630	.320

$h_1 = x_1 - x_0 = 0.12 - 0 = 0.12$
$h_2 = x_2 - x_1 = 0.22 - 0.12 = 0.1$
$h_3 = x_3 - x_2 = 0.32 - 0.22 = 0.1$

Sol: ∵ (h) is unequal for all table data

$$\therefore I = h_1 \frac{y_0+y_1}{2} + h_2 \frac{y_1+y_2}{2} + h_3 \frac{y_2+y_3}{2} + \frac{y_3+y_4}{2} \cdot h_4 + \dots + h_n \frac{y_{n-1}+y_n}{2}$$

$$= 0.12 \frac{0.2+0.97}{2} + 0.1 \frac{0.97+0.52}{2} + 0.1 \frac{0.52+0.43}{2} + 0.04 \frac{0.43+0.75}{2} + \dots + 0.1 \frac{0.63+.32}{2}$$

$I = \dots$

Ex.12: Solve the above Ex.11 by; Trapizoidal, Simpson's (1/3) & Simpson's (3/8).

Sol: In this case, we must check all the elements (h) for all data.

So, $h_1 = x_1 - x_0 = 0.12 - 0 \rightarrow h_1 = 0.12$ → Trapizoidal Rule.

$h_2 = x_2 - x_1 = 0.22 - 0.12 \rightarrow h_2 = 0.1$
$h_3 = x_3 - x_2 = 0.32 - 0.22 \rightarrow h_3 = 0.1$
] → Simpson's (1/3) Rule

$h_4 = x_4 - x_3 = 0.36 - 0.32 \rightarrow h_4 = 0.04$
$h_5 = x_5 - x_4 = 0.4 - 0.36 \rightarrow h_5 = 0.04$
$h_6 = x_6 - x_5 = 0.44 - 0.4 \rightarrow h_6 = 0.04$
] → Simpson's (3/8) Rule

$h_7 = x_7 - x_6 = 0.54 - 0.44 \rightarrow h_7 = 0.1$
$h_8 = x_8 - x_7 = 0.64 - 0.54 \rightarrow h_8 = 0.1$
] → Simpson's (1/3) Rule

$h_9 = x_9 - x_8 = 0.7 - 0.64 \rightarrow h_9 = 0.06$ → Trapizoidal Rule

$h_{10} = x_{10} - x_9 = 0.8 - 0.7 \rightarrow h_{10} = 0.1$ → Trapizoidal Rule

$$I_1 = h_1 \frac{y_0+y_1}{2} = 0.12 \frac{0.2+0.97}{2} \Rightarrow I_1 = 0.0702$$

$$I_2 = \frac{1}{3} h [y_0 + 4y_2 + y_3] = \frac{1}{3}(0.1)[0.97 + 4(0.52) + 0.43] \rightarrow I_2 = 0.116$$