Ch.4: Solution of Ordinary Differential Equs.

There are many different scientific problems, which leads to Ordinary Differential Equs.(O.D.E.) or Partial Differential Equs.(P.D.E.) their solutions are difficult, therefore we can solve it by “Numerical (approximation) Solution”.

The solution of (O.D.E.) is meaning find it’s function which satisfy (O.D.E.).

The general formula of (O.D.E.) first order is:

$$\frac{dy}{dx} = \bar{y} = f(x, y)$$
$$y(x_o) = y_o$$

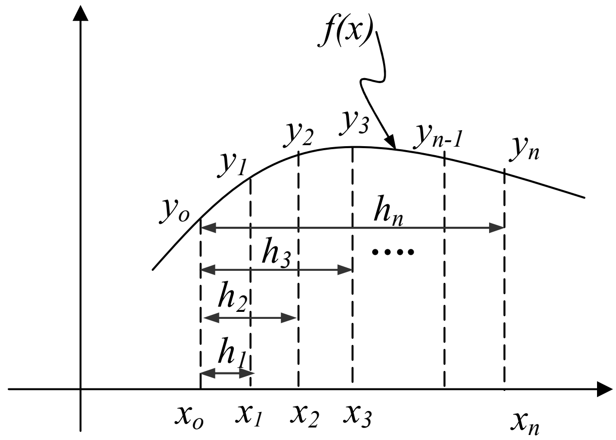
There are different types methods for solution of (O.D.E.):

1- Taylor Series Method
2- Euler Method
3- Modified Euler Method
4- Runge-Kutta Method

1- Taylor Series Method

The general formula of Taylor series is:

$$y_n = y_o + \frac{h_n}{1!}.y' + \frac{h_n^2}{2!}.y'' + \frac{h_n^3}{3!}.y''' + ... + \frac{h_n^c}{c!}.y^c$$



For easy solution, we can make a table to record the values of numerical & analytical.

Ex.1: Solve the (D.E.) $y' - x + y = 0$ by using “Taylor Series Method” when y(0)=1, x=0 (0.1) 0.4

Sol. $y(x_o) = y(0) = y_o=1$, x=0 (0.1) 0.4 → $x=x_o\ (h)\ x_n$

At $x_1=0.1$, $x_2=0.2$, $x_3=0.3$, $x_4=0.4$, we must find y_1, y_2, y_3, and y_4

$$y' = x - y \qquad \rightarrow y' = -1$$
$$y'' = 1 - y' = 1 - x + y \qquad \rightarrow y'' = 2$$
$$y''' = -1 + y' = -1 + x - y \qquad \rightarrow y''' = -2$$
$$y^{IV} = 1 - y' = 1 - x + y \qquad \rightarrow y^{IV} = 2$$

$$y_n = y_o + \frac{h_n}{1!}.y' + \frac{h_n^2}{2!}.y'' + \frac{h_n^3}{3!}.y''' + ... + \frac{h_n^c}{c!}.y^c$$

n=1 → $y_o=1$, $x_o=0$, $h_1=0.1$

$$y_n = y_o + \frac{h_1}{1}.y' + \frac{h_1^2}{2}.y'' + \frac{h_1^3}{3}.y''' + \frac{h_1^4}{4}.y^{IV}$$
$$y_1 = 1 + + \frac{0.1}{1}.(-1) + \frac{(0.1)^2}{2}.(2) + \frac{(0.1)^3}{3}.(-2) + \frac{(0.1)^4}{4}.(2)$$
$$y_1 = 0.9096$$

n=2 → $y_o=1$, $x_o=0$, $h_2=0.2$

$$y_n = y_o + \frac{h_2}{1}.y' + \frac{h_2^2}{2}.y'' + \frac{h_2^3}{3}.y''' + \frac{h_2^4}{4}.y^{IV}$$
$$y_2 = 1 + + \frac{0.2}{1}.(-1) + \frac{(0.2)^2}{2}.(2) + \frac{(0.2)^3}{3}.(-2) + \frac{(0.2)^4}{4}.(2)$$
$$y_2 = 0.8374$$

n=3 → $y_o=1$, $x_o=0$, $h_3=0.3$

$$y_n = y_o + \frac{h_3}{1}.y' + \frac{h_3^2}{2}.y'' + \frac{h_3^3}{3}.y''' + \frac{h_3^4}{4}.y^{IV}$$
$$y_3 = 1 + + \frac{0.3}{1}.(-1) + \frac{(0.3)^2}{2}.(2) + \frac{(0.3)^3}{3}.(-2) + \frac{(0.3)^4}{4}.(2)$$
$$y_3 = 0.7816$$

n=4 → $y_o=1$, $x_o=0$, $h_4=0.4$

$$y_n = y_o + \frac{h_4}{1}.y' + \frac{h_4^2}{2}.y'' + \frac{h_4^3}{3}.y''' + \frac{h_4^4}{4}.y^{IV}$$
$$y_4 = 1 + + \frac{0.4}{1}.(-1) + \frac{(0.4)^2}{2}.(2) + \frac{(0.4)^3}{3}.(-2) + \frac{(0.4)^4}{4}.(2)$$
$$y_4 = 0.7406$$

Ex.2: Solve the (D.E.) $y'x = 2y$ by using "Taylor Series Method" when y(0)=1, x=1 (0.2) 2

Sol. $y(x_o) = y(0) = y_o=1$, x=1 (0.2) 2 → $x=x_o\ (h)\ x_n$

At $x_1=1.2$, $x_2=1.4$, $x_3=1.6$, $x_4=1.8$, $x_5=2$, we must find y_1, y_2, y_3, y_4 and y_5

$y' = 2y/x \qquad \rightarrow y' = 2$

$y'' = 2y/x^2 \qquad \rightarrow y'' = 2$

$y''' = 0/x \qquad \rightarrow y''' = 0$

$y^{IV} = 0 \qquad \rightarrow y^{IV} = 0$

$y^{V} = 0 \qquad \rightarrow y^{V} = 0$

$$y_n = y_o + \frac{h_n}{1l}.y' + \frac{h_n^2}{2l}.y'' + \frac{h_n^3}{3l}.y''' + ... + \frac{h_n^c}{cl}.y^c$$

n=1 → $y_o=1$, $x_o=1$, $h_1=0.2$

$$y_n = y_o + \frac{h_1}{1}.y' + \frac{h_1^2}{2}.y'' + \frac{h_1^3}{3}.y''' + \frac{h_1^4}{4}.y^{IV} + \frac{h_1^5}{5}.y^V$$

$$y_1 = 1 + + \frac{0.2}{1}.(2) + \frac{(0.2)^2}{2}.(2) + \frac{(0.2)^3}{3}.(0) + \frac{(0.2)^4}{4}.(0) + \frac{(0.2)^5}{5}.(0)$$

$$y_1 = 1.44$$

n=2 → $y_o=1$, $x_o=1$, $h_2=0.4$

$$y_n = y_o + \frac{h_2}{1}.y' + \frac{h_2^2}{2}.y'' + \frac{h_2^3}{3}.y''' + \frac{h_2^4}{4}.y^{IV} + \frac{h_2^5}{5}.y^V$$

$$y_2 = 1 + + \frac{0.4}{1}.(2) + \frac{(0.4)^2}{2}.(2) + \frac{(0.4)^3}{3}.(0) + \frac{(0.4)^4}{4}.(0) + \frac{(0.4)^5}{5}.(0)$$

$$y_2 = 1.96$$

n=3 → $y_o=1$, $x_o=1$, $h_3=0.6$

$$y_n = y_o + \frac{h_3}{1}.y' + \frac{h_3^2}{2}.y'' + \frac{h_3^3}{3}.y''' + \frac{h_3^4}{4}.y^{IV} + \frac{h_3^5}{5}.y^V$$

$$y_3 = 1 + + \frac{0.6}{1}.(2) + \frac{(0.6)^2}{2}.(2) + \frac{(0.6)^3}{3}.(0) + \frac{(0.6)^4}{4}.(0) + \frac{(0.6)^5}{5}.(0)$$

$$y_3 = 2.56$$

n=4 → $y_o=1$, $x_o=1$, $h_4=0.8$

$$y_n = y_o + \frac{h_4}{1}.y' + \frac{h_4^2}{2}.y'' + \frac{h_4^3}{3}.y''' + \frac{h_4^4}{4}.y^{IV} + \frac{h_4^5}{5}.y^V$$

$$y_4 = 1 + + \frac{0.8}{1}.(2) + \frac{(0.8)^2}{2}.(2) + \frac{(0.8)^3}{3}.(0) + \frac{(0.8)^4}{4}.(0) + \frac{(0.8)^5}{5}.(0)$$

$$y_4 = 3.24$$

n=5 → $y_o=1$, $x_o=1$, $h_5=1$

$$y_n = y_o + \frac{h_5}{1}.y' + \frac{h_5^2}{2}.y'' + \frac{h_5^3}{3}.y''' + \frac{h_5^4}{4}.y^{IV} + \frac{h_5^5}{5}.y^{V}$$

$$y_5 = 1 + + \frac{1}{1}.(2) + \frac{(1)^2}{2}.(2) + \frac{(1)^3}{3}.(0) + \frac{(1)^4}{4}.(0) + \frac{(1)^5}{5}.(0)$$

$$y_5 = 4$$

H.W. By using "Taylor Series Method", solve the following (D.E.):

1- $y' = 2xy$ when y(0)=1, x=1 (0.2) 2

2- $y' - y - x = 0$ when y(1)=1, x=1 (0.1) 2

3- $y' - yx^2 = 0$ when y(0)=1, x=0 (0.5) 2

2- Euler (Simple Euler) Method:

In this method, we must provide "Initial value problem" & the solution is starting at this point and forward step by step in the range of the function.

$$y_{n+1} = y_n + h.f(x_n, y_n)$$

$$y_{n+1} = y_n + h.y'_n$$

Ex.3: Solve the (D.E.) $y' - x + y = 0$ by using "Euler Method" when y(0)=1, x=0 for x=0.1, 0.2, 0.3, 0.4, 0.5

Sol. $y(x_o) = y(0) = y_o=1$, $x=0\ (0.1)\ 0.4 \rightarrow x=x_o\ (h)\ x_n$

At $x_o=0$, $x_1=0.1$, $x_2=0.2$, $x_3=0.3$, $x_4=0.4$ & $x_5=0.5$ we must find y_1, y_2, y_3, y_4 and y_5

n=0 $\rightarrow y_o=1,\ x_o=0,\ h=0.1$

$$y_{n+1} = y_n + h.f(x_n, y_n)$$
$$y_1 = y_0 + h.f(x_0, y_0)$$
$$y_1 = 1 + 0.1(0 - 1)$$
$$y_1 = 0.9$$

n=1 $\rightarrow y_1=0.9,\ x_1=0.1,\ h=0.1$

$$y_{n+1} = y_n + h.f(x_n, y_n)$$
$$y_2 = y_1 + h.f(x_1, y_1)$$
$$y_2 = 0.9 + 0.1(0.1 - 0.9)$$
$$y_2 = 0.82$$

n=2 $\rightarrow y_2=0.82,\ x_2=0.2,\ h=0.1$

$$y_{n+1} = y_n + h.f(x_n, y_n)$$
$$y_3 = y_2 + h.f(x_2, y_2)$$
$$y_3 = 0.82 + 0.1(0.2 - 0.82)$$
$$y_3 = 0.758$$

n=3 $\rightarrow y_3=0.758,\ x_3=0.3,\ h=0.1$

$$y_{n+1} = y_n + h.f(x_n, y_n)$$
$$y_4 = y_3 + h.f(x_3, y_3)$$
$$y_4 = 0.758 + 0.1(0.3 - 0.758)$$
$$y_4 = 0.712$$

n=4 $\rightarrow y_4=0.758,\ x_4=0.4,\ h=0.1$

$$y_{n+1} = y_n + h.f(x_n, y_n)$$
$$y_5 = y_4 + h.f(x_4, y_4)$$
$$y_5 = 0.712 + 0.1(0.4 - 0.712)$$
$$y_5 = 0.680$$

Ex.4: Solve the (D.E.) $y' - x^2 - 4y = -0.5y$ by using "Euler Method" when y(0)=4, x=0,(0.05) 0.25

Sol. $y(x_o) = y(0) = y_o = 4$, x=0 (0.05) 0.2 → $x = x_o \ (h) \ x_n$

At $x_o=0$, $x_1=0.05$, $x_2=0.1$, $x_3=0.15$, $x_4=0.2$, $x_5=0.25$ we must find y_1, y_2, y_3, y_4 and y_5

n=0 → $y_o=4$, $x_o=0$, h=0.05

$$y_{n+1} = y_n + h.f(x_n, y_n)$$
$$y_1 = y_0 + h.f(x_0, y_0)$$
$$y_1 = 4 + 0.05[(0)^2 + 4.(0) - 0.5(4)]$$
$$y_1 = 3.9$$

n=2 → $y_1=3.9$, $x_1=0.05$, h=0.05

$$y_{n+1} = y_n + h.f(x_n, y_n)$$
$$y_2 = y_1 + h.f(x_1, y_1)$$
$$y_2 = 3.9 + 0.05[(0.05)^2 + 4.(0.05) - 0.5(3.9)]$$
$$y_2 = 3.81$$

n=3 → $y_2=3.81$, $x_2=0.1$, h=0.05

$$y_{n+1} = y_n + h.f(x_n, y_n)$$
$$y_3 = y_2 + h.f(x_2, y_2)$$
$$y_3 = 3.81 + 0.05[(0.1)^2 + 4.(0.1) - 0.5(3.81)]$$
$$y_3 = 3.73$$

n=4 → $y_3=3.73$, $x_3=0.15$, h=0.05

$$y_{n+1} = y_n + h.f(x_n, y_n)$$
$$y_4 = y_3 + h.f(x_3, y_3)$$
$$y_4 = 3.73 + 0.05[(0.15)^2 + 4.(0.15) - 0.5(3.73)]$$
$$y_4 = 3.67$$

n=5 → $y_4=3.67$, $x_4=0.2$, h=0.05

$$y_{n+1} = y_n + h.f(x_n, y_n)$$
$$y_5 = y_4 + h.f(x_4, y_4)$$
$$y_5 = 3.67 + 0.05[(0.2)^2 + 4.(0.2) - 0.5(3.67)]$$
$$y_5 = 3.62$$

Ex.5: If water is drained from a vertical cylinder tank by this equ. $dy/dt = -k\sqrt{y}$,k= constant (-0.06) & y= depth of water (m), t= time(min). Find the depth of water after (2min) of flow water. When initial water level (3m), step time (0.5 min) use Simple Euler Method.

Sol. $y(t_o) = y(0) = y_o=3$, $t=0\ (0.5)\ 2 \rightarrow t=t_o\ (\Delta t)\ t_n \leftrightarrow x=x_o\ (h)\ x_n$

At $x_o=0$, $x_1=0.5$, $x_2=1$, $x_3=1.5$, $x_4=2$ we must find y_1, y_2, y_3, and y_4 by Euler Method

n=0 $\rightarrow y_o=3, x_o=0, h=0.5$

$$y_{n+1} = y_n + h.f(x_n, y_n)$$
$$y_1 = y_0 + h.f(x_0, y_0)$$
$$y_1 = 3 + 0.5[-0.06(\sqrt{3})]$$
$$y_1 = 2.948\ m$$

n=1 $\rightarrow y_1=2.948, x_1=0.5 , h=0.5$

$$y_{n+1} = y_n + h.f(x_n, y_n)$$
$$y_2 = y_0 + h.f(x_0, y_0)$$
$$y_2 = 2.948 + 0.5[-0.06(\sqrt{2.948})]$$
$$y_2 = 2.896\ m$$

n=2 $\rightarrow y_2=2.896, x_2=1 , h=0.5$

$$y_{n+1} = y_n + h.f(x_n, y_n)$$
$$y_3 = y_0 + h.f(x_0, y_0)$$
$$y_3 = 2.896 + 0.5[-0.06(\sqrt{2.896})]$$
$$y_3 = 2.845\ m$$

n=3 $\rightarrow y_3=2.845, x_3=1.5 , h=0.5$

$$y_{n+1} = y_n + h.f(x_n, y_n)$$
$$y_4 = y_0 + h.f(x_0, y_0)$$
$$y_4 = 2.845 + 0.5[-0.06(\sqrt{2.845})]$$
$$y_4 = 2.794\ m \qquad at \quad t = 2\min$$

H.W. Solve the above example by using Taylor series method**.**

3- Modified Euler Method, Euler's Trapezoidal Method, Predictor Corrected Method

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$$

The estimation value *$f(x_{n+1} + y_{n+1})$* find from last "**Euler Method"**

Ex.6: Solve the (D.E.) $y' = xy$ by using "Modified Euler Method" when y(1)=1, x=1,(0.2), 2

Sol. $y(x_o) = y(1) = y_o=1$, x=1,(0.2), 2 → $x=x_o$ (h) x_n

At $x_o=1$, $x_1=1.2$, $x_2=1.4$, $x_3=1.6$, $x_4=1.8$, & $x_5=2$, we must find y_1, y_2, y_3, y_4, and y_5

n=0 → $y_o=1$, $x_o=1$, h=0.2

$$y_{n+1} = y_n + h.f(x_n, y_n)$$
$$y_1 = y_0 + h.f(x_0, y_0)$$
$$y_1 = 1 + 0.2(1*1)$$
$$y_1 = 1.2 \quad \textit{Estimation value}$$

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$$
$$y_1 = y_o + \frac{h}{2}[f(x_o, y_o) + f(x_1, y_1)] \qquad ;\ x_1=1.2,\ y_1=1.2$$
$$y_1 = 1 + \frac{0.2}{2}[(1*1) + (1.2*1.2)]$$
$$y_1 = 1.244 \quad \textit{Corrected value}$$

n=1 → $y_1=1.244$, $x_1=1.2$, h=0.2

$$y_{n+1} = y_n + h.f(x_n, y_n)$$
$$y_2 = y_1 + h.f(x_1, y_1)$$
$$y_2 = 1.244 + 0.2(1.2*1.244)$$
$$y_2 = 1.542 \quad \textit{Estimation value}$$

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$$
$$y_2 = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2)] \qquad ;\ x_2=1.4,\ y_2=1.542$$
$$y_2 = 1.244 + \frac{0.2}{2}[(1.2*1.244) + (1.4*1.542)]$$
$$y_2 = 1.609 \quad \textit{Corrected value}$$

n=2 → $y_2=1.609$, $x_2=1.4$, h=0.2

$$y_{n+1} = y_n + h.f(x_n, y_n)$$
$$y_3 = y_2 + h.f(x_2, y_2)$$
$$y_3 = 1.609 + 0.2(1.4*1.609)$$
$$y_3 = 2.059 \quad \textit{Estimation value}$$

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$$

$$y_3 = y_2 + \frac{h}{2}[f(x_2, y_2) + f(x_3, y_3)]$$; x_3=1.6, y_3=2.059

$$y_3 = 1.609 + \frac{0.2}{2}[(1.4 * 1.069) + (1.6 * 2.059)]$$

$$y_3 = 2.163 \qquad \textit{Corrected value}$$

n=3 → y_3=2.855, x_3=1.6, h=0.2

$$y_{n+1} = y_n + h.f(x_n, y_n)$$

$$y_4 = y_3 + h.f(x_3, y_3)$$

$$y_4 = 2.263 + 0.2(1.6 * 2.163)$$

$$y_4 = 2.855 \qquad \textit{Estimation value}$$

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$$

$$y_4 = y_3 + \frac{h}{2}[f(x_3, y_3) + f(x_4, y_4)]$$; x_4=1.8, y_4=2.855

$$y_4 = 2.163 + \frac{0.2}{2}[(1.6 * 2.163) + (1.8 * 2.855)]$$

$$y_4 = 3.023 \qquad \textit{Corrected value}$$

n=4 → y_4=3.022, x_3=1.8, h=0.2

$$y_{n+1} = y_n + h.f(x_n, y_n)$$

$$y_5 = y_4 + h.f(x_4, y_4)$$

$$y_5 = 3.023 + 0.2(1.8 * 3.023)$$

$$y_5 = 4.111 \qquad \textit{Estimation value}$$

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$$

$$y_5 = y_4 + \frac{h}{2}[f(x_4, y_4) + f(x_5, y_5)]$$; x_5=2, y_5=4.111

$$y_5 = 3.023 + \frac{0.2}{2}[(1.8 * 3.023) + (2 * 4.111)]$$

$$y_5 = 4.390 \qquad \textit{Corrected value}$$

Ex.7: Solve the (D.E.) $y' = x - y$ by using "Euler Trapezoidal Method" when y(0)=1, x=0.1, 0.2, 0.3, 0.4

Sol. $y(x_o) = y(0) = y_o$=1 , x=0.1, 0.2, 0.3, 0.4 → h=0.1

At x_1=0.1, x_1=0.2, x_3=0.3, & x_4=0.4, we must find y_1, y_2, y_3, and y_4

n=0 → y_o=1, x_o=0, h=0.1

$$y_{n+1} = y_n + h.f(x_n, y_n)$$
$$y_1 = y_0 + h.f(x_0, y_0)$$
$$y_1 = 1 + 0.1(0-1)$$
$$y_1 = 0.9 \quad Estimation \ \ value$$

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$$
$$y_1 = y_o + \frac{h}{2}[f(x_o, y_o) + f(x_1, y_1)] \qquad ;\ x_1=0.2,\ y_1=0.9$$
$$y_1 = 1 + \frac{0.1}{2}[(0.1*1) + (0.2*0.9)]$$
$$y_1 = 0.91 \quad Corrected \ value$$

n=1 → y_1=0.91, x_1=0.1, h=0.1

$$y_{n+1} = y_n + h.f(x_n, y_n)$$
$$y_2 = y_1 + h.f(x_1, y_1)$$
$$y_2 = 0.91 + 0.1(0.1 - 0.91)$$
$$y_2 = 0.829 \quad Estimation \ \ value$$

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$$
$$y_2 = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2)] \qquad ;\ x_2=0.2,\ y_2=0.829$$
$$y_2 = 0.91 + \frac{0.1}{2}[(0.1*0.91) + (0.2*0.829)]$$
$$y_2 = 0.838 \quad Corrected \ value$$

n=2 → y_2=0.838, x_2=0.2, h=0.1

$$y_{n+1} = y_n + h.f(x_n, y_n)$$
$$y_3 = y_2 + h.f(x_2, y_2)$$
$$y_3 = 0.838 + 0.1(0.2 - 0.838)$$
$$y_3 = 0.774 \quad Estimation \ \ value$$

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$$
$$y_3 = y_2 + \frac{h}{2}[f(x_2, y_2) + f(x_3, y_3)] \qquad ;\ x_3=0.3,\ y_3=0.774$$
$$y_3 = 0.838 + \frac{0.1}{2}[(0.2*0.838) + (0.3*0.774)]$$
$$y_3 = 0.782 \quad Corrected \ value$$

n=3 → y_3=0.782, x_3=0.3, h=0.1

$$y_{n+1} = y_n + h.f(x_n, y_n)$$

$$y_4 = y_3 + h.f(x_3, y_3)$$

$$y_4 = 0.782 + 0.1(0.3 - 0.782)$$

$$y_4 = 0.734 \qquad Estimation\ \ value$$

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$$

$$y_4 = y_3 + \frac{h}{2}[f(x_3, y_3) + f(x_4, y_4)]$$; x_4=0.4, y_4=0.734

$$y_4 = 0.782 + \frac{0.1}{2}[(0.3 * 0.782) + (0.4 * 0.734)]$$

$$y_4 = 0.714 \qquad Corrected\ value$$

4- Runge-Kutta Method

This method is more advanced from others methods that can be classified into two types:

A- Second order Runge-Kutta formula

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

$$k_1 = h.f'(x_n, y_n)$$
$$k_2 = h.f'(x_n + h, y_n + k_1)$$

Ex.8: Solve the (D.E.) $y' = x.y$ by using "Second order Method" when y(1)=1, x=1.1, 1.2,

Sol. $y(x_o) = y(1) = y_o=1$, x=1.1, 1.2, → h=0.1

n=0 → $y_o=1$, $x_o=1$, h=0.1

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$
$$k_1 = h.f'(x_n, y_n) = h.f'(x_0, y_0) = 0.1\,f'(1,1) = 0.1(1*1) \rightarrow k_1 = 0.1$$
$$k_2 = h.f'(x_n + h, y_n + k_1) = h.f'(x_0 + h, y_0 + k_1) = 0.1\ f'(1+0.1, 1+0.1) = 0.1\ f'(1.1, 1.1)$$
$$k_2 = 0.1(1.1*1.1)$$
$$k_2 = 0.121$$
$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2) = 1 + \frac{1}{2}(0.1 + 0.121)$$
$$y_1 = 1.1105$$

n=1 → $y_1=1.1105$, $x_1=1.1$, h=0.1

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$
$$k_1 = h.f'(x_n, y_n) = h.f'(x_1, y_1) = 0.1\,f'(1.1, 1.1105) = 0.1(1.1*1.1105) \rightarrow k_1 = 0.1221$$
$$k_2 = h.f'(x_n + h, y_n + k_1) = h.f'(x_1 + h, y_1 + k_1) = 0.1\ f'(1.1+0.1, 1.1105+0.1221) = 0.1\ f'(1.2, 1.2326)$$
$$k_2 = 0.1(1.2*1.2326)$$
$$k_2 = 0.148$$
$$y_2 = y_1 + \frac{1}{2}(k_1 + k_2) = 1 + \frac{1}{2}(0.1221 + 0.148)$$
$$y_2 = 1.1350$$

H.W. Solve

1- above Ex. y(1)=1, x=1,(0.1), 1.6

2- $y' = x+y$ y(1)=1, x=1,(0.2), 2

3- $y' - yx^2 = 0$ y(0)=1, x=0,(0.2), 1

Ex.9: Find a velocity after (min) by this given equ. $du/dt = g + (c/m).u$ where are (g, c, m) constants equal to (9.81, 12.5, 68). At initial time (0) the velocity is (0 m/min), take step time (1min). Use 2nd Order method.

B- Fourth order Runge-Kutta formula

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = h.f'(x_n, y_n)$$
$$k_2 = h.f'(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$$
$$k_3 = h.f'(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$$
$$k_4 = h.f'(x_n + h, y_n + k_3)$$

<u>Ex.8:</u> Solve the (D.E.) $y' = x - y$ by using "forth order Method" when y(0)=1, x=0.1, 0.2, 0.3, 0.4

<u>Sol.</u> $y(x_o) = y(0) = y_o=1$, x=0.1, 0.2, 0.3, 0.4 , → h=0.1

<u>n=0</u> → $y_o=1$, $x_o=0$, h=0.1

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = h.f'(x_n, y_n) = h.f'(x_0, y_0) = 0.1\ f'(0,1) = 0.1\ (0-1)$$
$$k_1 = -0.1$$
$$k_2 = h.f'(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}) = h.f'(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.1 f'(0 + \frac{0.1}{2}, 1 - \frac{0.1}{2}) = 0.1[(0 + \frac{0.1}{2}) - (1 - \frac{0.1}{2})]$$
$$k_2 = -0.09$$
$$k_3 = h.f'(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}) = h.f'(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}) = 0.1 f'(0 + \frac{0.1}{2}, 1 - \frac{0.09}{2}) = 0.1[(0 + \frac{0.1}{2}) - (1 - \frac{0.09}{2})]$$
$$k_3 = -0.0905$$
$$k_4 = h.f'(x_n + h, y_n + k_3) = h.f'(x_0 + h, y_0 + k_3) = 0.1.f'(0 + 0.1, 1 - 0.0905) = 0.1[(0 + 0.1) - (1 - 0.0905)]$$
$$k_4 = -0.0809$$
$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1 + \frac{1}{6}(-0.1 - 2(0.09) - 2(0.0905) - 0.0809)$$
$$y_1 = 0.9096$$

<u>n=1</u> → y_1=0.9096, x_1=0.1, h=0.1

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = h.f'(x_n, y_n) = h.f'(x_1, y_1) = 0.1\ f'(0.1, 0.9096) = 0.1\ (0.1 - 0.9096)$$
$$k_1 = -0.08096$$
$$k_2 = h.f'(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}) = h.f'(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = 0.1 f'(0.1 + \frac{0.1}{2}, 0.9096 - \frac{0.08096}{2})$$
$$= 0.1[(0.1 + \frac{0.1}{2}) - (0.9096 - \frac{0.08096}{2})]$$
$$k_2 = -0.072$$
$$k_3 = h.f'(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}) = h.f'(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}) = 0.1 f'(0.1 + \frac{0.1}{2}, 0.9096 - \frac{0.072}{2})$$
$$= 0.1[(0.1 + \frac{0.1}{2}) - (0.9096 - \frac{0.072}{2})]$$
$$k_3 = -0.0723$$
$$k_4 = h.f'(x_n + h, y_n + k_3) = h.f'(x_1 + h, y_1 + k_3) = 0.1.f'(0.1 + 0.1, 0.9096 - 0.0723)$$
$$= 0.1[(0.1 + 0.1) - (0.9096 - 0.0723)]$$
$$k_4 = -0.0637$$
$$y_2 = y_2 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.9096 + \frac{1}{6}(-0.08096 - 2(0.072) - 2(0.0723) - 0.0637)$$
$$y_2 = 0.9277$$

<u>n=2</u> → y_2=0.9277, x_2=0.2, h=0.1 By same steps above, we can find y_3

<u>n=3</u> → y_3=????? , x_3=0.3, h=0.1 And also by same steps above, we can find y_4

H.W. Solve by 2nd & 4th order

1- above Ex. y(1)=1, x=1,(0.1), 1.4

2- $y' = x + y$ y(1)=1, x=1,(0.2), 2

3- $y' - yx^2 = 0$ y(0)=1, x=0,(0.2), 1

Summary
Solution of Ordinary Differential Equs.

1- Taylor Series Method

$$y_n = y_o + \frac{h_n}{1i}.y' + \frac{h_n^2}{2i}.y'' + \frac{h_n^3}{3i}.y''' + ... + \frac{h_n^c}{ci}.y^c$$

2- Euler (Simple Euler) Method

$$y_{n+1} = y_n + h.f(x_n, y_n)$$

$$y_{n+1} = y_n + h.y_n'$$

3- Modified Euler Method, Euler's Trapezoidal Method, Predictor Corrected Method

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})]$$

The estimation value $f(x_{n+1}+ y_{n+1})$ find from last "**Euler Method"**

4- Runge-Kutta Method

A- Second order

$$y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$$

$$k_1 = h.f'(x_n, y_n)$$

$$k_2 = h.f'(x_n + h, y_n + k_1)$$

B- Fourth order

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = h.f'(x_n, y_n)$$

$$k_2 = h.f'(x_n + \frac{h}{2}, y_n + \frac{k_1}{2})$$

$$k_3 = h.f'(x_n + \frac{h}{2}, y_n + \frac{k_2}{2})$$

$$k_4 = h.f'(x_n + h, y_n + k_3)$$

For General $y(x_o) = y(0) = y_o=1$, $x_o=0$, $x=x_o$ (h) x_n → for example x=0 (0.1) 0.4