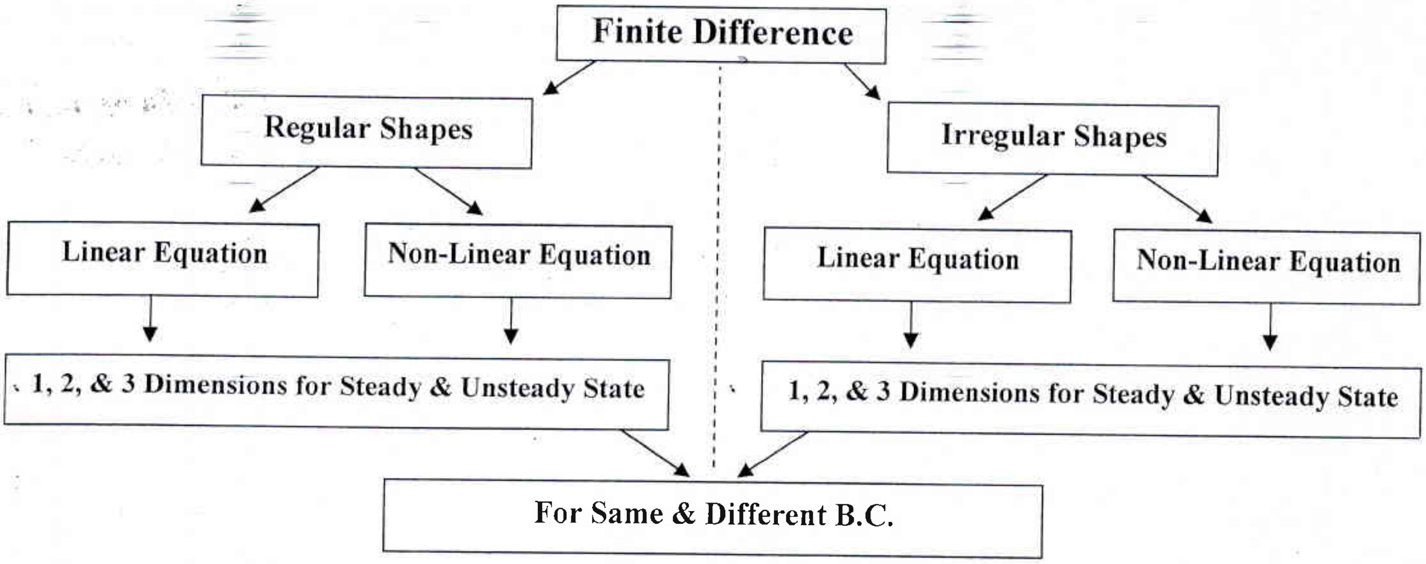
Ch.5: Finite Difference Methods for Solution of Differential Equation (O.D.E.) & partial Differential Equs. (P.D.E.)



There are many engineering problems in formula (P.D.E.). (O.D.E.) it's direct solution is very difficult but it's convert to simple equs. system which can solve by easy numerical method

(P.D.E.) can be classified into:

I → Elliptic Equ. → Laplace's Equ. $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$

→ Poisson's Equ. $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f(x,y)$

II → Parabolic Equ. $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$

III → Hyperbolic Equ.

Second Order: $\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x^2}$

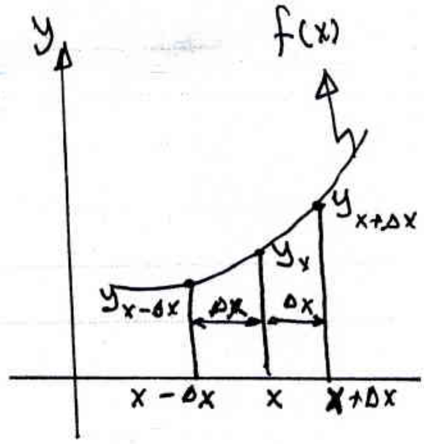
first Order: $\frac{\partial u}{\partial t}=\frac{\partial u}{\partial x}$

A-1-Regular Shapes, Linear Equs., One dimensions :

Steady state:

By Taylor Series Forword :-

$$y_{x+\Delta x} = y_x + \Delta x\,\bar{y} + \frac{\Delta x^2}{2!}\,\bar{\bar{y}} + \frac{\Delta x^3}{3!}\,y''' + \cdots$$



By Taylor Series Backword :-

$$y_{x-\Delta x} = y_x - \Delta x\,\bar{y} + \frac{\Delta x^2}{2!}\,y'' - \frac{\Delta x^3}{3!}\,y''' + \cdots$$

Now; Find $(\bar{y})$ for forword, backword & center Exprission

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* By Forword :-

$$y_{x+\Delta x} = y_x + \Delta x\,\bar{y} \,\Big|\, + \frac{\Delta x^2}{2!}\,y'' + \frac{\Delta x^3}{3!}\,y''' + \cdots$$

cutting

$$\left[\ \bar{y} = \frac{dy}{dx} = \frac{y_{x+\Delta x} - y_x}{\Delta x}\ \right];\quad E_O\ \ \Delta x$$

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* By Backword :-

$$y_{x-\Delta x} = y_x - \Delta x\,\bar{y} \,\Big|\, + \frac{\Delta x^2}{2!}\,y'' - \frac{\Delta x^3}{3!}\,y''' + \cdots$$

cutting

$$\left[\ \bar{y} = \frac{dy}{dx} = \frac{y_x - y_{x-\Delta x}}{\Delta x}\ \right];\quad E_O\ \ \Delta x$$

* By Center Expression :-

Forword Exp. − Backword Exp. →
Tayler S.   Tayler S.

$$y_{x+\Delta x} = \cancel{y_x} + \Delta x\, y' \qquad \text{Forward Tayler Exp.}$$

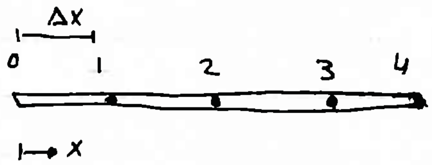
$$y_{x-\Delta x} = \cancel{y_x} - \Delta x\, y' \qquad \text{Backword Tayler Exp.}$$

بالطرح

$$y_{x+\Delta x} - y_{x-\Delta x} = 2\Delta x\, y'$$

$$\left[y' = \frac{dy}{dx} = \frac{y_{x+\Delta x} - y_{x-\Delta x}}{2\Delta x} \right];$$

Ex. 1: solve by finite difference $[\,T' + 2T = 2x\,]$ & then solve the system matrix by one-Method of solution set simultaneous Eq.
$T(0) = 50°c$, $T(1) = 50°c$



Sol: Nodes ①, ② & ③ solve by F.D.
Center expression

$$y' = \frac{y_{x+\Delta x} - y_{x-\Delta x}}{2\Delta x} \quad \text{sub. in O.D.E.} \rightarrow$$

$$\left[\frac{T_{x+\Delta x} - T_{x-\Delta x}}{2\Delta x} + 2T_x = 2x \right] \text{General formula} , \; \Delta x = \frac{L}{M-1} = \frac{1}{4-1} = 0.$$

No. Nodes

Node ①
$x = 0.25$
$\Delta x = 0.25$

$$\frac{T_2 - T_0}{2(0.25)} + 2T_1 = 2(0.25) \qquad *(0.25)\,2$$

$$T_2 - 50 + 4(0.25)\,T_1 = 4(0.25)^2$$

$$T_2 + T_1 = 50.25 \quad \text{-----} \; ①$$

Node ②
$x = 0.5$
$\Delta x = 0.25$

$$\frac{T_3 - T_1}{2(0.25)} + 2T_2 = 2(0.5) \qquad * \; 2(0.25)$$

$$T_3 - T_1 + 4(0.25)\,T_2 = 4(0.25)(0.5)$$

Node ④ $\qquad \dfrac{T_5 - T_3}{2\Delta x} + T_4 = -200(0.6) \qquad *(0.2)2$

$x = 0.6$

$\Delta x = 0.2$

$$T_5 - T_3 + (0.2)2\, T_4 = -200(0.6) * 2(0.2)$$

$$T_5 - T_3 + 0.4\, T_4 = -48 \quad --- (*)$$

at ⑤ $q = 0 \rightarrow \dfrac{\partial T}{\partial x} = 0 \rightarrow \dfrac{T_5 - T_4}{\Delta x} = 0 \rightarrow T_5 = T_4$ sub. in (*)

$$-T_3 + 1.4\, T_4 = -48 \quad --- ③$$

So, $\begin{bmatrix} T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} 0.4 & 1 & 0 \\ -1 & 0.4 & 1 \\ 0 & -1 & 1.4 \end{bmatrix} = \begin{bmatrix} 4 \\ -32 \\ -48 \end{bmatrix}$

H.W. Solve Ex.2 when B.C

① $T(1) \rightarrow \dfrac{\partial T}{\partial x} = 50$

$T(0) = 20°c$

② $T(1), T(0) \rightarrow \dfrac{\partial T}{\partial x} = 0$

H.W. 1) Solve Ex.1 when Number of Nodes (8) = M.

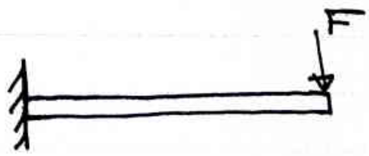
2) Solve Ex.2 when $\Delta x = 1$ Unit length & length of the rod is (7) Unit length.

3) Solve Ex.1 when $\Delta x = 1$ Unit length & length of the rod is (5) Unit length.

4) Solve Ex.2 when $M = 8$.

5) Solve by F.D. to find distribution of deflection along the bar for this Equ.

$$\frac{dy}{dx} + 2y = 0$$



fig(a)

at B.C. ⓐ $y(0) = 0$, $y(1) = 20$, $M = 7$] fig(a)

B.C ⓑ $y(1) = y(0) = 0$, $M = 7$] fig.(b)

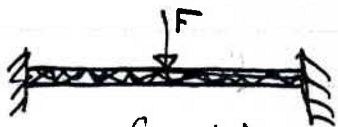


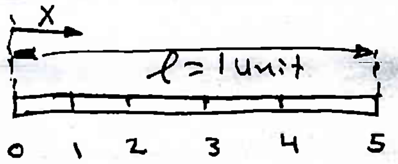
fig.(b)

B.C ⓒ $y(0) = y(1) = 0$, $M = 6$

Fig.(C)

Ex.3: Solve the following Equ. $\ddot{y} + 2\dot{y} + y = 3x$ by finite Difference

$y_0 = 100$, $y_5 = 20$ (diffirent B.C.)



Sol. For Nodes ① ② ③ ④ Solve by Center expression

$$\ddot{y} = \frac{y_{x+\Delta x} - 2y_x + y_{x-\Delta x}}{\Delta x^2} \quad \& \quad \dot{y} = \frac{y_{x+\Delta x} - y_{x-\Delta x}}{2\Delta x}$$

Now; Sub. in above O.D.E. ⇒

$$\left[\frac{y_{x+\Delta x} - 2y_x + y_{x-\Delta x}}{\Delta x^2} + 2\,\frac{y_{x+\Delta x} - y_{x-\Delta x}}{2\Delta x} + y_x = 3x \right] \text{General formula}$$

Node ① $x = \Delta x = 0.2$, $y_0 = 100$

$$\frac{y_2 - 2y_1 + y_0}{(0.2)^2} + \frac{y_2 - y_0}{(0.2)} + y_1 = 3(0.2) \qquad *(0.2)^2$$

$$y_2 - 2y_1 + 100 + 0.2y_2 - 20 + 0.2^2(y_1) = 3(0.2)^3$$

$$1.2y_2 - 1.96y_1 = -79.97 \quad ---- ①$$

Node ② $x = 0.4$

$$\frac{y_3 - 2y_2 + y_1}{(0.2)^2} + \frac{y_3 - y_1}{0.2} + y_2 = 3(0.4) \qquad *(0.2)^2$$

$$y_3 - 2y_2 + y_1 + 0.2y_3 - 0.2y_1 + 0.2^2\,y_2 = 3(0.4)(0.2)^2$$

$$1.2y_3 - 1.96y_2 + 0.8y_1 = 0.048 \quad ---- ②$$

Node ③ $x = 0.6$

$$\frac{y_4 - 2y_3 + y_2}{(0.2)^2} + \frac{y_4 - y_2}{0.2} + y_3 = 3(0.6) \qquad *(0.2)^2$$

$$y_4 - 2y_3 + y_2 + 0.2y_4 - 0.2y_2 + 0.2^2\,y_3 = 3(0.6)(0.2)^2$$

$$1.2y_4 - 1.96y_3 + 0.8y_2 = 0.072 \quad ---- ③$$

Node ④ $x = 0.8$

$$\frac{y_5 - 2y_4 + y_3}{(0.2)^2} + \frac{y_5 - y_3}{0.2} + y_4 = 3(0.8) \qquad * (0.2)^2$$

$$\cancel{y_5}^{\,0} - 2y_4 + y_3 + 0.2\cancel{y_5}^{\,0} - 0.2y_3 + 0.2^2 y_4 = 3(0.8)(0.2)^2$$

$$-1.96y_4 + 0.8y_3 = 0.096 \quad ---- ④$$

So,

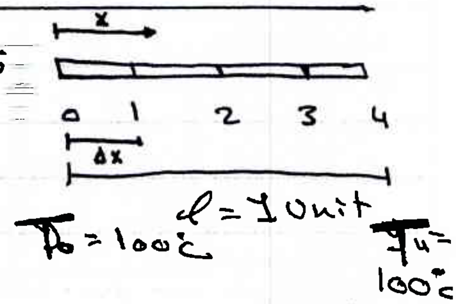
$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} * \begin{bmatrix} -1.96 & 1.2 & 0 & 0 \\ 0.8 & -1.96 & 1.2 & 0 \\ 0 & 0.8 & -1.96 & 1.2 \\ 0 & 0 & 0.8 & -1.96 \end{bmatrix} = \begin{bmatrix} -79.97 \\ 0.048 \\ 0.072 \\ 0.096 \end{bmatrix}$$

H.W. solve when B.C.

① $y(0) = y(1) = 0$

② $y(0) = y(1) = 50$

Ex.2: Solve the following Equ. $\frac{\partial^2 T}{\partial x^2} + 100 = 0$;



(Same B.C.)

Sol. For Nodes ①②③ Solve by Center exp.

$$\bar{\bar{y}} = \bar{\bar{T}} = \frac{\partial^2 T}{\partial x^2} = \frac{T_{x+\Delta x} - 2T_x + T_{x-\Delta x}}{\Delta x^2}$$

Now; sub. in above P.D.E. ⇒

Node ① $x = 0.25$, $\Delta x = 0.25$

$$\frac{T_2 - 2T_1 + T_0}{\Delta x^2} + 100 = 0$$

$$\frac{T_2 - 2T_1 + T_0}{(0.25)^2} + 100 = 0 \qquad * (0.25)^2$$

$$T_2 - 2T_1 + 100 + (0.25)^2 100 = 0$$

$$T_2 - 2T_1 = -106.25 \quad ---- ①$$

Calculate y''' forward, backword, & center exprission:

* By forward

$$y''' = \frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) \; ; \text{Let } \psi = \frac{d^2y}{dx^2}$$

$$\frac{d^3y}{dx^3} = \frac{d\psi}{dx} = \frac{\psi_{x+\Delta x} - \psi_x}{\Delta x} \quad ---(1)$$

$$\psi_{x+\Delta x} = \left.\frac{d^2y}{dx^2}\right)_{x+\Delta x} = \frac{y_{x+\Delta x} - 2y_{x+2\Delta x} + y_{x+3\Delta x}}{\Delta x^2} \quad ---(2)$$

$$\psi_x = \left.\frac{d^2y}{dx^2}\right)_x = \frac{y_x - 2y_{x+\Delta x} + y_{x+2\Delta x}}{\Delta x^2} \quad ---(3)$$

sub. (2)(3) in (1) $\Rightarrow$

$$\left[\frac{d\psi}{dx} = \frac{d^3y}{dx^3} = \frac{y_{x+3\Delta x} - 3y_{x+2\Delta x} - y_x + 3y_{x+\Delta x}}{\Delta x^3}\right] ; E \circ \Delta x$$

* By backword

$$\frac{d^3y}{dx^3} = \frac{d\psi}{dx} = \frac{\psi_x - \psi_{x-\Delta x}}{\Delta x} \quad ---(1)$$

$$\psi_{x-\Delta x} = \left.\frac{d^2y}{dx^2}\right)_{x-\Delta x} = \frac{y_{x-3\Delta x} - 2y_{x-2\Delta x} + y_{x+\Delta x}}{\Delta x^2} \quad ---(2)$$

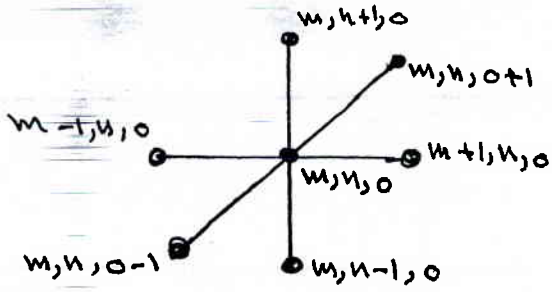
$$\psi_x = \left.\frac{d^3y}{dx^3}\right)_x = \frac{y_{x-2\Delta x} - 2y_{x-\Delta x} + y_x}{\Delta x^2} \quad ---(3)$$

sub. (2)(3) in (1) $\Rightarrow$

$$\left[\frac{d\psi}{dx} = \frac{d^3y}{dx^3} = \frac{y_x + 3y_{x+2\Delta x} - 3y_{x-\Delta x} - y_{x-3\Delta x}}{\Delta x^3}\right] ; E \circ \Delta x$$

A-3- Regular shapes, Linear Equ, three dim., steady state

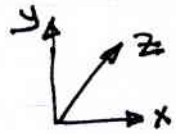
Let $W = f(x, y, z)$



For Center expressions:-

$$\frac{\partial w}{\partial z} = \frac{W_{m,n,o+1} - W_{m,n,o-1}}{2\Delta z}$$

$$\frac{\partial w}{\partial y} = \frac{W_{m,n+1,o} - W_{m,n-1,o}}{2\Delta y}$$



$$\frac{\partial w}{\partial x} = \frac{W_{m+1,n,o} - W_{m-1,n,o}}{2\Delta x}$$

$$\frac{\partial^2 w}{\partial z^2} = \frac{W_{m,n,o+1} - 2W_{m,n,o} + W_{m,n,o-1}}{\Delta z^2}$$

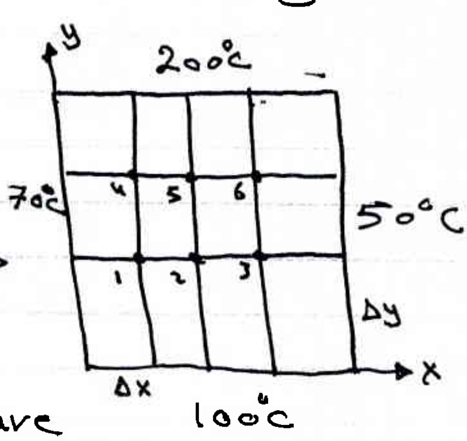
$$\frac{\partial^2 w}{\partial y^2} = \frac{W_{m,n+1,o} - 2W_{m,n,o} + W_{m,n-1,o}}{\Delta y^2}$$

$$\frac{\partial^2 w}{\partial x^2} = \frac{W_{m+1,n,o} - 2W_{m,n,o} + W_{m-1,n,o}}{\Delta x^2}$$

Note: The boundary condition can be write by:

$T(x, y)$ for $\Delta x = \Delta y = 1$ unit

$$\left.\begin{aligned} &T(x,0) = 100^\circ c \;\&\; T(x,1) = 200^\circ \\ &T(1,y) = 50^\circ c \;\&\; T(0,y) = 70^\circ c \end{aligned}\right] \Rightarrow$$



II) You must take min. No. nodes are (4), if there is n't limited in question; in Two dim.
& 3D; 1D we take min. No. nodes are (3).

Ex.8: Solve the equ. $\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$; which subjected to the conditions are showed in fig.(1) when $\Delta x = \Delta y = 1$

(Laplace's Equ.)

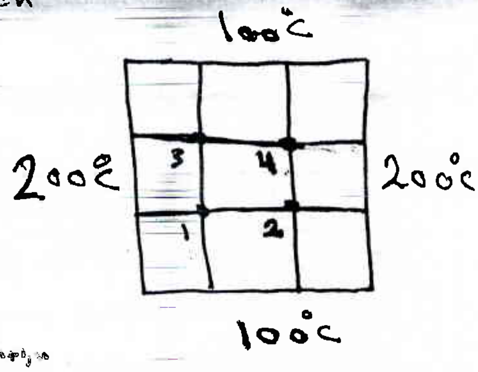


fig.(1)

Sol. By center expression :-

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{m+1,n} - 2T_{m,n} + T_{m-1,n}}{\Delta x^2}$$

$$\frac{\partial^2 T}{\partial y^2} = \frac{T_{m,n+1} - 2T_{m,n} + T_{m,n-1}}{\Delta y^2}$$

So, subst in above equ. ⇒

$$\frac{T_{m+1,n} - 2T_{m,n} + T_{m-1,n}}{\Delta x^2} + \frac{T_{m,n+1} - 2T_{m,n} + T_{m,n-1}}{\Delta y^2} = 0 \quad ; \Delta x = \Delta y = 1 \Rightarrow$$

$$\left[T_{m+1,n} + T_{m-1,n} + T_{m,n+1} + T_{m,n-1} - 4T_{m,n} = 0 \right] \text{ General formula}$$

Node ①: $T_2 + 200 + T_3 + 100 - 4T_1 = 0$

$$T_3 + T_2 - 4T_1 = -300 \quad \text{---} ①$$

Node ②: $T_1 + 200 + T_4 + 100 - 4T_2 = 0$

$$T_4 + T_1 - 4T_2 = -300 \quad \text{---} ②$$

Node ③: $T_4 + 200 + T_1 + 100 - 4T_3 = 0$

$$T_4 + T_1 - 4T_3 = -300 \quad \text{---} ③$$

Node ④: $T_3 + 200 + T_2 + 100 - 4T_4 = 0$

$$T_3 + T_2 - 4T_4 = -300 \quad \text{---} ④$$

So, arranged the above equs.(1, 2, 3, & 4) in matrix system ⇒

Node ④: $20 + T_3 + 10 + T_2 - 4T_4 = -(1)^2(1)(0.5)^2$

$x = 1$

$y = 1$ $\qquad T_3 + T_2 - 4T_4 = -30.25$ --- (4)

Node ⑤: $T_6 + 0 + T_3 + 10 - 4T_5 = -(0.5)^2(0.75)(0.5)^2$

$x = 0.5$

$y = 0.75$ $\qquad T_6 + T_3 - 4T_5 = -10.046$ --- (5)

Node ⑥: $T_5 + 20 + 10 + T_4 - 4T_6 = (1)^2(0.75)(0.5)^2$

$x = 1$

$y = 0.75$ $\qquad T_5 + T_4 - 4T_6 = -30.187$ --- (6)

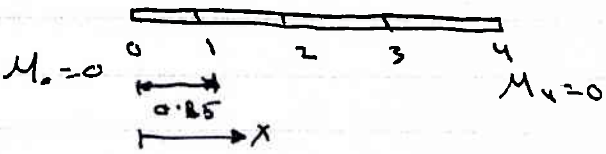
So, arrenged the above equs (1 → 6) in matrix system ⇒

$$\begin{bmatrix} -4 & 1 & 1 & 0 & 0 & 0 \\ 1 & -4 & 0 & 1 & 0 & 0 \\ 1 & 0 & -4 & 1 & 1 & 0 \\ 0 & 1 & 1 & -4 & 0 & 0 \\ 0 & 0 & 1 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 1 & -4 \end{bmatrix} \cdot \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \end{bmatrix} = \begin{bmatrix} -30.031 \\ -50.125 \\ -0.0625 \\ -30.25 \\ -10.046 \\ -30.187 \end{bmatrix}$$

H.W. Solve when $\Delta x = \Delta y = 1$ & No. of nodes are (9); 3 nodes in x-axis and 3 nodes in y-axis.

Ex.8: Solve the distribution of bending moment in a beam subjected to loading by a distribution load [W(x)] per unit length as showed in fig.(3)

$$\frac{d^2M}{dx^2} = W(x)$$



$W(x) = \sin \pi x$

Sol/ By center expression:-

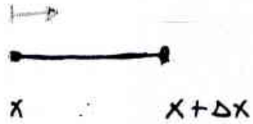
$$\frac{d^2M}{dx^2} = \frac{M_{m+1} - 2M_m + M_{m-1}}{\Delta x^2}$$ sub. in above equ.

$$\frac{M_{m+1} - 2M_m + M_{m-1}}{\Delta x^2} = \sin(\pi \cdot x) * \Delta x^2$$

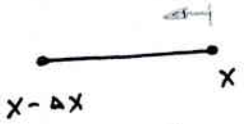
$$\left[M_{m+1} - 2M_m + M_{m-1} = \Delta x^2 \cdot \sin \pi x \right]$$ General formula

Summary

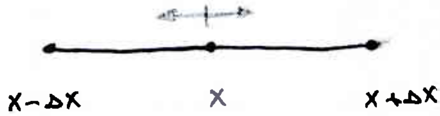
Forword : $$y' = \frac{y_{x+\Delta x} - y_x}{\Delta x}$$



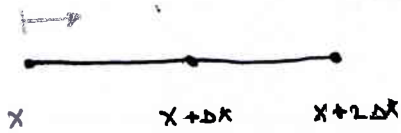
Backword : $$y' = \frac{y_x - y_{x-\Delta x}}{\Delta x}$$



Center : $$y' = \frac{y_{x+\Delta x} - y_{x-\Delta x}}{2\Delta x}$$

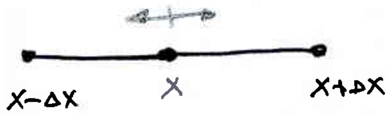


Forword : $$y'' = \frac{y_{x+2\Delta x} - 2y_{x+\Delta x} + y_x}{\Delta x^2}$$

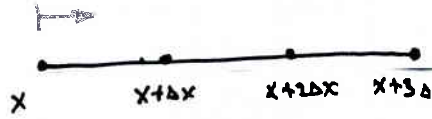


Backword : $$y'' = \frac{y_{x-2\Delta x} - 2y_{x-\Delta x} + y_x}{\Delta x^2}$$

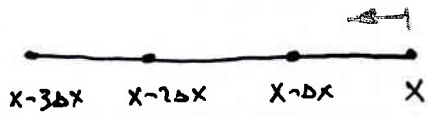
Center : $$y'' = \frac{y_{x+\Delta x} - 2y_x + y_{x-\Delta x}}{\Delta x^2}$$



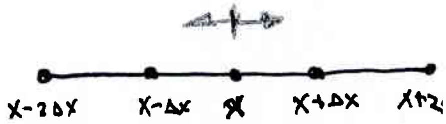
Forword : $$y''' = \frac{y_{x+3\Delta x} - 3y_{x+2\Delta x} + 3y_{x+\Delta x} - y_x}{\Delta x^3}$$



Backword : $$y''' = \frac{y_x + 3y_{x-2\Delta x} - y_{x-3\Delta x} - 3y_{x-\Delta x}}{\Delta x^3}$$



Center : $$y''' = \frac{y_{x+2\Delta x} - 2y_{x+\Delta x} - y_{x-2\Delta x} + 2y_{x-\Delta x}}{\Delta x^3}$$



A-4- Regular shapes, Linear equ., 1, 2, 3D, Unsteady state:

In Unsteady state problems; the time (t) in depended in solving O.D.E & P.D.E by forword expression solution for time.

$$\frac{dT}{dt} = \frac{T^{n+1} - T^{n}}{\Delta t}$$; (n) is iteration of time.

$$\frac{\dot{q}}{K} + \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$ (Unsteady, 3D, Heat Conduction) with heat generation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$ (Unsteady, 2D, heat Conduction) without heat generation

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}$$ (Unsteady, 1D, heat conduction) parabolic equ.

So;

$$\left[\frac{T^{n}_{m+1,n,o} - 2T^{n}_{m,n,o} + T^{n}_{m-1,n,o}}{\Delta x^2} + \frac{T^{n}_{m,n+1,o} - 2T^{n}_{m,n,o} + T^{n}_{m,n-1,o}}{\Delta y^2} + \frac{T^{n}_{m,n,o+1} - 2T^{n}_{m,n,o} + T^{n}_{m,n,o-1}}{\Delta z^2} = \frac{1}{\alpha}\frac{T^{n+1}_{m,n,o} - T^{n}_{m,n,o}}{\Delta t} \right]$$ General formula

In this case, we need 3 B. → 2 B.C. , 1 initial condition] 1 dim.

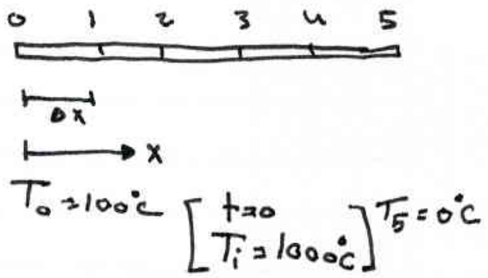
5 B. → 4 B.C , 1 initial Condition] 2 dim.

7 B. → 6 B.C. , 1 initial Condition] 3 dim.

* $\alpha = \frac{K}{\rho . C_p}$

Ex.9: Solve the following equ. $T'' = \frac{1}{\alpha}(\partial T/\partial t)$ for B.C. as $T(x,t)$, $T(x,0) = 1000°c$, $T(0,t) = 100°c$, $T(1,t) = 0°c$ when $\alpha = (K/\rho . c_p) = 50$ & Δt = nothing or 0.0002 unit time. at time $(t) = 0.0032$ unit time

Sol: Center expression & unsteady state



$$\frac{\partial^2 T}{\partial x^2} = \frac{T^n_{m+1} - 2T^n_m + T^n_{m-1}}{\Delta x^2} \text{ ; sub. in above equ.} \Rightarrow$$

$$\frac{T^n_{m+1} - 2T^n_m + T^n_{m-1}}{\Delta x^2} = \frac{1}{\alpha}\,\frac{T^{n+1}_m - T^n_m}{\Delta t} \quad * \Delta x^2 \Rightarrow$$

$$T^n_{m+1} - 2T^n_m + T^n_{m-1} = \frac{\Delta x^2}{\Delta t . \alpha}\left(T^{n+1}_m - T^n_m\right)$$

$$\boxed{T^n_{m+1} + T^n_{m-1} + \left(\frac{\Delta x^2}{\Delta t . \alpha} - 2\right)T^n_m = \frac{\Delta x^2}{\Delta t . \alpha}\, T^{n+1}_m} \text{ General formula}$$

$$\frac{\Delta x^2}{\Delta t . \alpha} - 2 \geq 0 \Rightarrow \text{I)} \quad \frac{\Delta x^2}{\Delta t . \alpha} - 2 = 0 \quad \text{when } \Delta t = \text{Nothing;}$$

it's not given in Ques.

$$\frac{\Delta x^2}{\Delta t . \alpha} = 2 \text{ ; } \Delta x = 0.2, \alpha = 50$$

$\Delta t = 0.0004$ unit time ; sub. in G.F. $\Rightarrow$

$$T^n_{m+1} + T^n_{m-1} = 2\,T^{n+1}_m \Rightarrow \boxed{T^{n+1}_m = \frac{T^n_{m+1} + T^n_{m-1}}{2}} \text{ --- (I)}$$

$t = \Delta t \Rightarrow$ Node(1)

0.0004 $\quad T^1_1 = \frac{1000+100}{2} = 550°c$

Node(2)

$T^1_2 = \frac{1000+1000}{2} = 1000°c$

Node(3)

$T^1_3 = \frac{1000+1000}{2} = 1000°c$

Node(4)

$T^1_4 = \frac{1000+0}{2} = 500°c$

Nodes / Time	0	1	2	3	4	5
$t=\Delta t=0$	100	1000	1000	1000	1000	0
$t=1\Delta t$	100	550	1000	1000	500	0
$t=2\Delta t$	100	550	775	750	500	0
$t=3\Delta t$	100	437.5	650	637.5	375	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮
$t=0.0032$						

$t=2\Delta t \Rightarrow$ Node ① $T_1^2=(1000+100)/2 \Rightarrow T_1^2=550\,°C$
0.0008

Node ② $T_2^2=(1000+550)/2 \Rightarrow T_2^2=775\,°C$

Node ③ $T_3^2=(1000+500)/2 \Rightarrow T_3^2=750\,°C$

Node ④ $T_4^2=(1000+0)/2 \Rightarrow T_4^2=500\,°C$

$t=3\Delta t \Rightarrow$ Node ① $T_1^3=(775+100)/2 \Rightarrow T_1^3=437.5\,°C$
0.0012

Node ② $T_2^3=(750+550)/2 \Rightarrow T_2^3=650\,°C$

Node ③ $T_3^3=(500+775)/2 \Rightarrow T_3^3=637.5\,°C$

Node ④ $T_4^3=(0+750)/2 \Rightarrow T_4^3=375\,°C$

when $\frac{\Delta x^2}{\Delta t.\alpha}-2>0 \Rightarrow$ II) at $\Delta t=0.0002$ unit time (is given in Q.)

$\frac{\Delta x^2}{\Delta t.\alpha}=4$ sub. in G.F. $\Rightarrow$

$$\left[\, T_{m+1}^n+T_{m-1}^n+2\,T_m^n=4\,T_m^{n+1} \,\right] \quad --- \quad \text{(II)}$$

$t=\Delta t \Rightarrow$ Node ① $T_1^1=(1000+100+2*1000)/4 \Rightarrow T_1^1=775\,°C$
0.0002

Node ② $T_2^1=(1000+1000+2*1000)/4 \Rightarrow T_2^1=1000\,°C$

Node ③ $T_3^1=(1000+1000+2*1000)/4 \Rightarrow T_3^1=1000\,°C$

Node ④ $T_4^1=(0+1000+2*1000)/4 \Rightarrow T_4^1=750\,°C$

$t=2\Delta t \Rightarrow$ Node ① $T_1^2=(1000+100+2*775) \Rightarrow T_1^2=662.5\,°C$
0.0004

Node ② $T_2^2=(1000+775+2*1000) \Rightarrow T_2^2=943.75\,°C$

Node ③ $T_3^2 = (1000+775+2*1000) \Rightarrow T_3^2 = 937.5\ ^\circ C$

Node ④ $T_4^2 = (0+1000+2*750) \Rightarrow T_4^2 = 625\ ^\circ C$

Time \ Node	0	1	2	3	4	5
t=0	100	1000	1000	1000	1000	0
t=Δt	100	775	1000	1000	750	0
t=2Δt	100	662.5	943.75	937.5	625	0
t=3Δt	100	⋮	⋮	⋮	⋮	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮
t=0.0032						

Ex.18 Solve the following equ. $\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=40\frac{\partial T}{\partial t}$ for B.C. as $T(x,y,t)$ when $T(0,y,t)=100$

$T(1,y,t)=100$ & $T(x,0,t)=T(x,1,t)=500$ & $T(x,y,0)=0$

$\Delta x=\Delta y=0.25$, after $3\Delta t$.

Sol/ $\frac{\partial^2 T}{\partial x^2}=\frac{T^n_{m+1,n}-2T^n_{m,n}+T^n_{m-1,n}}{\Delta x^2}$; $\frac{\partial^2 T}{\partial y^2}=\frac{T^n_{m,n+1}-2T^n_{m,n}+T^n_{m,n-1}}{\Delta y^2}$

$\frac{\partial T}{\partial t}=\frac{T^{n+1}_{m,n}-T^n_{m,n}}{\Delta t}$ sub. in P.D.E. $\Rightarrow$

$$\frac{T^n_{m+1,n}-2T^n_{m,n}+T^n_{m-1,n}}{\Delta x^2}+\frac{T^n_{m,n+1}-2T^n_{m,n}+T^n_{m,n-1}}{\Delta y^2}=40\frac{T^{n+1}_{m,n}-T^n_{m,n}}{\Delta t} \quad *\Delta x^2$$

$$T^n_{m+1,n}-2T^n_{m,n}+T^n_{m-1,n}+T^n_{m,n+1}-2T^n_{m,n}+T^n_{m,n-1}=\frac{40\Delta x^2}{\Delta t}\left(T^{n+1}_{m,n}-T^n_{m,n}\right)$$

$$\left[T^n_{m+1,n}+T^n_{m-1,n}+T^n_{m,n+1}+T^n_{m,n-1}+\left(\frac{40\Delta x^2}{\Delta t}-4\right)T^n_{m,n}=\frac{40\Delta x^2}{\Delta t}T^{n+1}_{m,n}\right] \text{G.F.}$$

$\frac{40\Delta x^2}{\Delta t}-4 \geq 0 \Rightarrow \frac{40\Delta x^2}{\Delta t}-4=0 \Rightarrow \frac{40\Delta x^2}{\Delta t}=0$ sub. in G.F. $\Rightarrow$

$$\left[\left(T^n_{m+1,n}+T^n_{m-1,n}+T^n_{m,n+1}+T^n_{m,n-1}\right)/4=T^{n+1}_{m,n}\right]$$

& $\frac{40\Delta x^2}{\Delta t}-4=0 \Rightarrow \Delta t=0.625$ unit time

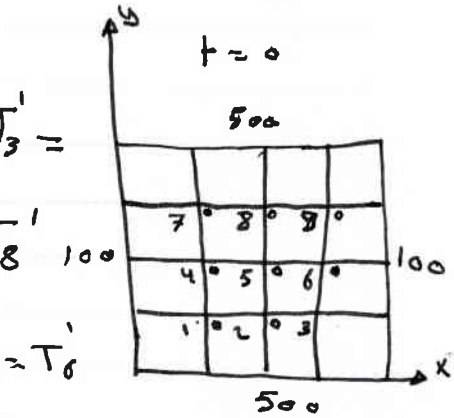
$t = \Delta t \Rightarrow$ Node ① $(100 + T_2 + T_4 + 500)/4 = T_1'$
0.625

$T_1' = (100 + 0 + 0 + 500)/4 \Rightarrow T_1' = 150°C = T_3' = T_7' = T_9'$

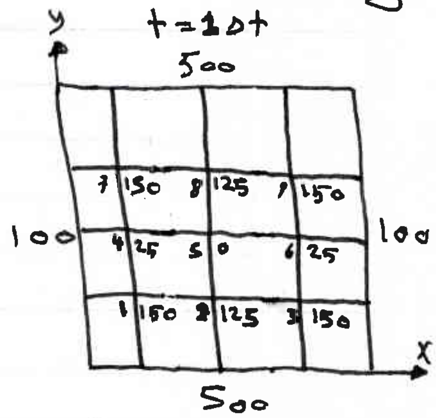
Node ② $T_2' = (0 + 0 + 0 + 500)/4 \Rightarrow T_2' = 125°C = T_8'$

Node ④ $T_4' = (100 + 0 + 0 + 0)/4 \Rightarrow T_4' = 25°C = T_6'$

Node ⑤ $T_5' = (0 + 0 + 0 + 0)/4 \Rightarrow T_5' = 0°C$



By Symmetry

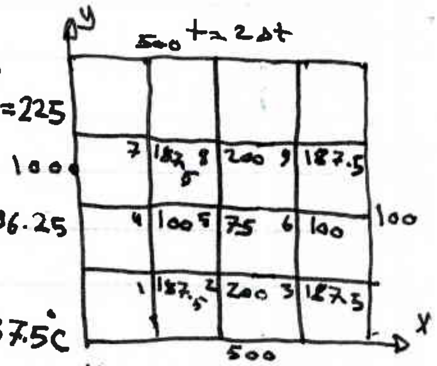


$t = 2\Delta t \Rightarrow$ Node ① $T_1^2 = (100 + 500 + 125 + 25)/4 \Rightarrow T_1^2 = 187.5°C = T_3^2 = T_7^2 = T_9^2$
1.25

Node ② $T_2^2 = (500 + 150 + 0 + 150)/4 \Rightarrow T_2^2 = 200°C = T_8^2$

Node ④ $T_4^2 = (100 + 150 + 150 + 0)/4 \Rightarrow T_4^2 = 100°C = T_6^2$

Node ⑤ $T_5^2 = (125 + 125 + 25 + 25)/4 \Rightarrow T_5^2 = 75°C$

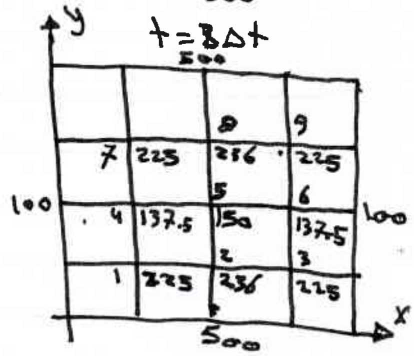


$t = 3\Delta t \Rightarrow$ Node ① $\Rightarrow T_1^3 = (100 + 500 + 100 + 200)/4 \Rightarrow T_1^3 = T_3^3 = T_7^3 = T_9^3 = 225$
1.875

Node ② $\Rightarrow T_2^3 = (187.5 + 187.5 + 500 + 75)/4 \Rightarrow T_2^3 = T_8^3 = 236.25$

Node ④ $\Rightarrow T_4^3 = (100 + 187.5 + 187.5 + 75)/4 \Rightarrow T_4^3 = T_6^3 = 137.5°C$

Node ⑤ $\Rightarrow T_5^3 = (200 + 200 + 100 + 100)/4 \Rightarrow T_5^3 = 150°C$



H.W. Solve for Same P.D.E. but when

$T(0,y,t) = 50°C$
$T(1,y,t) = 100°C$

$T(x,0,t) = 25°C$
$T(x,1,t) = 75°C$

& $\Delta x = \Delta y = 0.25$

after $3\Delta t$

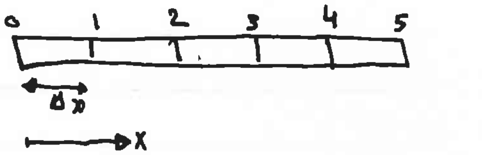
$T(x,y,0) = 0°C$

A-5- Regular shapes, Non-linear equ., 1,2,3D, steady state:

When $K = f(X \text{ or } T) \Rightarrow K = K_o(1+\beta T) \Rightarrow$ Let $K = 320(1+0.01T)$

Ex.1:

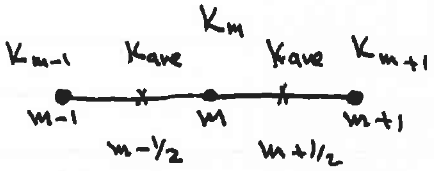
$$\frac{d}{dx}\left(K\frac{dT}{dx}\right) = -x^2 \text{ ; when } K = 320(1+0.01T) \text{ , } \Delta x = 0.2$$



$T_0 = 0°c$ $T_5 = 100°c$

Sol: Let $\psi = K\dfrac{dT}{dx}$ sub. in above equ. $\Rightarrow$

$$\left(\frac{d\psi}{dx}\right)_m = -x^2$$



$$\frac{\psi_{m+1/2} - \psi_{m-1/2}}{\Delta x} = -x^2 \quad \text{--- (1) Center Exp.}$$

$$\psi_{m+\frac{1}{2}} = \left(K\frac{dT}{dx}\right)_{m+\frac{1}{2}} = K_{m+\frac{1}{2}}\frac{T_{m+1} - T_m}{\Delta x} \quad \text{--- (2) Forword Exp.}$$

$$\psi_{m-\frac{1}{2}} = \left(K\frac{dT}{dx}\right)_{m-\frac{1}{2}} = K_{m-1/2}\frac{T_m - T_{m-1}}{\Delta x} \quad \text{--- (3) Backword Exp.}$$

So, sub. (2),(3) in (1) $\Rightarrow$

$$\frac{K_{m+1/2}(T_{m+1} - T_m) - K_{m-1/2}(T_m - T_{m-1})}{\Delta x^2} = -x^2 \qquad * \Delta x^2 \Rightarrow$$

$$\boxed{K_{m+\frac{1}{2}}T_{m+1} + K_{m-1/2}T_{m-1} - (K_{m+1/2} + K_{m-1/2})T_m = -x^2\Delta x^2} \text{ G.F.}$$

Node(1): $K_{3/2}T_2 + \cancel{K_{1/2}T_0}^{\,0} - (K_{3/2} + K_{1/2})T_1 = -(0.2)^2(0.2)^2$ --- (1)

Node(2): $K_{5/2}T_3 + K_{3/2}T_1 - (K_{5/2} + K_{1/2})T_2 = -(0.4)^2(0.2)^2$ --- (2)

Node(3): $K_{7/2}T_4 + K_{5/2}T_2 - (K_{7/2} + K_{5/2})T_3 = -(0.6)^2(0.2)^2$ --- (3)

Node(4): $K_{9/2}T_5 + K_{7/2}T_3 - (K_{9/2} + K_{7/2})T_4 = -(0.8)^2(0.2)^2$

$$K_{7/2}\,T_3 - (K_{9/2} + K_{7/2})\,T_4 = -(0.8)^2(0.2)^2 - 100\,K_{9/2} \qquad ---(4)$$

$$\begin{bmatrix} -(K_{3/2}+K_{1/2}) & K_{3/2} & 0 & 0 \\ K_{3/2} & -(K_{5/2}+K_{1/2}) & K_{5/2} & 0 \\ 0 & K_{5/2} & -(K_{7/2}+K_{5/2}) & K_{7/2} \\ 0 & 0 & K_{7/2} & -(K_{9/2}+K_{7/2}) \end{bmatrix} * \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} -0.0016 \\ -0.0064 \\ -0.0144 \\ -0.0256\,K_{9/2} \end{bmatrix}$$

Now, find the value of "K" for all Nodes by assumption
$T_1 = T_2 = T_3 = T_4 = \text{zero} \Rightarrow$

$$K_{1/2} = 320\left(1 + 0.01\,\frac{T_0 + T_1}{2}\right) = 320\left(1 + 0.01\,\frac{0+0}{2}\right) \Rightarrow K_{1/2} = 320$$

$$K_{3/2} = 320\left(1 + 0.01\,\frac{T_1 + T_2}{2}\right) = 320\left(1 + 0.01\,\frac{0+0}{2}\right) \Rightarrow K_{3/2} = 320$$

$$K_{5/2} = 320\left(1 + 0.01\,\frac{T_2 + T_3}{2}\right) = 320\left(1 + 0.01\,\frac{0+0}{2}\right) \Rightarrow K_{5/2} = 320$$

$$K_{7/2} = 320\left(1 + 0.01\,\frac{T_3 + T_4}{2}\right) = 320\left(1 + 0.01\,\frac{0+0}{2}\right) \Rightarrow K_{7/2} = 320$$

$$K_{9/2} = 320\left(1 + 0.01\,\frac{T_4 + T_5}{2}\right) = 320\left(1 + 0.01\,\frac{0+100}{2}\right) \Rightarrow K_{9/2} = 480$$

Sub. the value of "K" in above matrix $\Rightarrow$

$$\begin{bmatrix} -640 & 320 & 0 & 0 \\ 320 & -640 & 320 & 0 \\ 0 & 320 & -640 & 320 \\ 0 & 0 & 320 & -\ 0 \end{bmatrix} * \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} -0.0016 \\ -0.0064 \\ -0.0144 \\ -12.288 \end{bmatrix}$$

H.W $\rightarrow$ Solve above Ex.7 when $K = K_o(1 + \beta T)$; $K_o = 54$ & $\beta = 0.01$
at (1) $\Delta x = 0.25$ U.L. (2) $\Delta x = 0.125$ Unit Length

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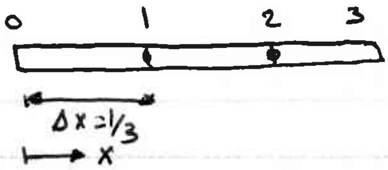
Ex.8) Solve the $\frac{\partial}{\partial x}\left(K \frac{dT}{dx}\right) = -10X$; $T(X) \Rightarrow T(0)=0°C$ & $T(1)=0°C$

$\Delta X = 1/3$, $K = K_o(1+\beta T)$ at $K_o = 1$ & $\beta = 0.1$.

Sol) Let $\psi = K\frac{\partial T}{\partial x} \Rightarrow \left(\frac{\partial \psi}{\partial x}\right)_m = -10X$



$T_o=0°c$, $K=(1+0.1T)$, $T_o=0°c$

$$\psi_{m+1/2} = \left(K\frac{\partial T}{\partial X}\right)_{m+1/2} = K_{m+1/2}\frac{T_{m+1}-T_m}{\Delta X} \quad ---(2)$$

$$\psi_{m+1/2} = \left(K\frac{\partial T}{\partial X}\right)_{m-1} = K_{m-1/2}\frac{T_m-T_{m-1}}{\Delta X} \quad ---(3)$$

$$\frac{\psi_{m+1/2}-\psi_{m-1/2}}{\Delta X} = -10X \quad ---(1) \text{ Center Exp.}$$

Sub. (2)(3) in (1) $\Rightarrow$

$$-10X = \frac{\partial}{\partial X}\left(K\frac{\partial T}{\partial X}\right) \Rightarrow \frac{K_{m+1/2}(T_{m+1}-T_m) - K_{m-1/2}(T_m - T_{m-1})}{\Delta X^2} = -10X$$

$$\left[K_{m+1/2}T_{m+1} + K_{m-1/2}T_{m-1} - (K_{m+1/2}+K_{m-1/2})T_m = -\Delta X^2 \times 10\right] \text{ G.F.}$$

Node(1) $K_{3/2}T_2 + K_{1/2}\cancel{T_0}^{\,0} - (K_{3/2}+K_{1/2})T_1 = -(1/3)(1/3)(10) \quad ---(1)$

Node(2) $K_{5/2}\cancel{T_3}^{\,0} + K_{3/2}T_1 - (K_{5/2}+K_{3/2})T_2 = -(1/3)^2(2/3)(10)$

$$\begin{bmatrix} -(K_{3/2}+K_{1/2}) & K_{3/2} \\ K_{3/2} & -(K_{5/2}+K_{3/2}) \end{bmatrix} \cdot \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} -0.37 \\ -0.73 \end{bmatrix}$$

* Now, find the value of "K" for all Nodes by assumption $T_1 = T_2 = 0$

$$K_{1/2} = 1+0.1\left(\frac{T_0+T_1}{2}\right) = 1+0.1\left(\frac{0+0}{2}\right) \Rightarrow K_{1/2} = 1$$

$$K_{3/2} = 1+0.1\left(\frac{T_1+T_2}{2}\right) = 1+0.1\left(\frac{0+0}{2}\right) \Rightarrow K_{3/2} = 1$$

$$K_{5/2} = 1+0.1\left(\frac{T_2+T_3}{2}\right) = 1+0.1\left(\frac{0+0}{2}\right) \Rightarrow K_{5/2} = 1$$

Sub. the value of "K" in above matrix ⇒

$$\begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \cdot \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} -0.37 \\ -0.74 \end{bmatrix} \xrightarrow{\text{Solving}} T_1 = 0.5^\circ C \ \& \ T_2 = 0.616^\circ C$$

* Now, Assume $T_1 = 0.5^\circ C$ & $T_2 = 0.616^\circ C$ To find new "K" ⇒

$$K_{1/2} = 1 + 0.1\left(\frac{0 + 0.5}{2}\right) \Rightarrow K_{1/2} = 1.025$$

$$K_{3/2} = 1 + 0.1\left(\frac{0.616 + 0.5}{2}\right) \Rightarrow K_{3/2} = 1.055$$

$$K_{5/2} = 1 + 0.1\left(\frac{0 + 0.616}{2}\right) \Rightarrow K_{5/2} = 1.03$$

Sub. in the above matrix ⇒

$$\begin{bmatrix} -2.080 & 1 \\ 1 & -2.085 \end{bmatrix} \cdot \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} -0.37 \\ -0.74 \end{bmatrix} \xrightarrow{\text{Solving}} T_1 = 0.453^\circ C \ \& \ T_2 = 0.572^\circ C$$

* and also Assume $T_1 = 0.453^\circ C$ & $T_2 = 0.572^\circ C$ find new "K" ⇒

$$K_{1/2} = 1 + 0.1\left(\frac{0 + 0.453}{2}\right) \Rightarrow K_{1/2} = 1.0226$$

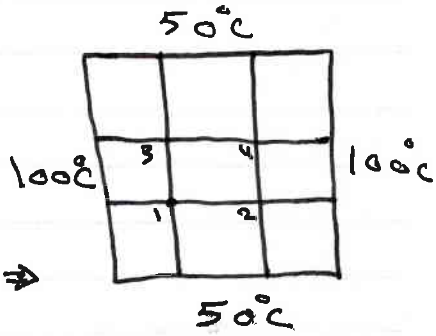
$$K_{3/2} = 1 + 0.1\left(\frac{0.572 + 0.453}{2}\right) \Rightarrow K_{3/2} = 1.0512$$

$$K_{5/2} = 1 + 0.1\left(\frac{0 + 0.572}{2}\right) \quad K_{5/2} = 1.0286$$

Sub. in above matrix ⇒

$$\begin{bmatrix} -2.073 & 1 \\ 1 & -2.079 \end{bmatrix} \cdot \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} -0.37 \\ -0.74 \end{bmatrix} \xrightarrow{\text{Solving}} T_1 = \qquad \& \ T_2 =$$

Ex/B: The plate show in fig.(5) has a variable thermal Conductivity $K = 0.1T + 200$. Find the Temperature distribution through the plate by solving $\frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial T}{\partial y}\right) = 0$ when $\Delta x = \Delta y = 1/3$



Sol: $\psi = K\frac{\partial T}{\partial x} \Rightarrow \left(\frac{\partial \psi}{\partial x}\right)_{m,n}$

$\Phi = K\frac{\partial T}{\partial y} \Rightarrow \left(\frac{\partial \Phi}{\partial y}\right)_{m,n}$ sub. in P.D.E. $\Rightarrow$

$$\left(\frac{\partial \psi}{\partial x} + \frac{\partial \Phi}{\partial y}\right)_{m,n} = 0 .$$

$$\frac{\psi_{m+1/2,n} - \psi_{m-1/2,n}}{\Delta x} + \frac{\Phi_{m,n+1/2} - \Phi_{m,n-1/2}}{\Delta y} = 0 \quad \text{--- (1) Center Exp.}$$

$$\psi_{m+1/2,n} = \left(K\frac{\partial T}{\partial x}\right)_{m+1/2,n} = K_{m+1/2,n}\frac{T_{m+1,n} - T_{m,n}}{\Delta x} \quad \text{--- (2) Forward Exp.}$$

$$\psi_{m-1/2,n} = \left(K\frac{\partial T}{\partial x}\right)_{m-1/2,n} = K_{m-1/2,n}\frac{T_{m,n} - T_{m-1,n}}{\Delta x} \quad \text{--- (3) Backword Exp.}$$

$$\Phi_{m,n+1/2} = \left(K\frac{\partial T}{\partial y}\right)_{m,n+1/2} = K_{m,n+1/2}\frac{T_{m,n+1} - T_{m,n}}{\Delta y} \quad \text{--- (4) Forword Exp.}$$

$$\Phi_{m,n-1/2} = \left(K\frac{\partial T}{\partial y}\right)_{m,n-1/2} = K_{m,n-1/2}\frac{T_{m,n} - T_{m,n-1}}{\Delta y} \quad \text{--- (5) Backword Exp.}$$

Now, sub. (2)(3)(4)(5) in (1) $\Rightarrow$

$$\frac{K_{m+1/2,n}(T_{m+1,n} - T_{m,n}) - K_{m-1/2,n}(T_{m,n} - T_{m-1,n})}{\Delta x^2} +$$

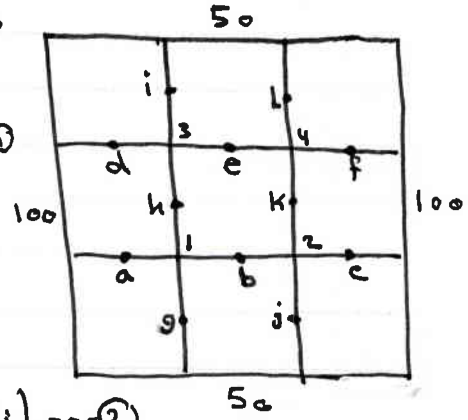
$$\frac{K_{m,n+1/2}(T_{m,n+1} - T_{m,n}) - K_{m,n-1/2}(T_{m,n} - T_{m,n-1})}{\Delta y^2} = 0$$

$* \Delta x^2$ & arranged it $\Rightarrow$

$$\left[K_{m+\frac{1}{2},n}T_{m+1,n}+K_{m-\frac{1}{2},n}T_{m-1,n}+K_{m,n+\frac{1}{2}}T_{m,n+1}+K_{m,n-\frac{1}{2}}T_{m,n-1}-\left(K_{m+\frac{1}{2},n}+K_{m-\frac{1}{2},n}+K_{m,n+\frac{1}{2}}+K_{m,n-\frac{1}{2}}\right)T_{m,n}=0\right] \quad G.F.$$

Node ①

$$K_bT_2+K_a(100)+K_g(50)+K_hT_3-(K_a+K_b+K_g+K_h)T_1=0$$



$$K_bT_2+K_hT_3-(K_a+K_b+K_g+K_h)T_1=-(100K_a+50K_g) \text{ --- ①}$$

Node ②

$$K_c(100)+K_bT_1+K_kT_4+K_j(50)-(K_c+K_b+K_k+K_j)T_2=0$$

$$K_bT_1+K_kT_4-(K_c+K_b+K_k+K_j)T_2=-(100K_c+50K_j) \text{ --- ②}$$

Node ③

$$K_eT_4+K_d(100)+K_i(50)+K_hT_1-(K_e+K_d+K_i+K_h)T_3=0$$

$$K_eT_4+K_hT_1-(K_e+K_d+K_i+K_h)T_3=-(100K_d+50K_i) \text{ --- ③}$$

Node ④

$$K_f(100)+K_eT_3+K_1(50)+K_kT_2-(K_f+K_e+K_1+K_k)T_4=0$$

$$K_eT_3+K_kT_2-(K_f+K_e+K_1+K_k)T_4=-(100K_f+50K_1) \text{ --- ④}$$

$$\begin{bmatrix} -(K_a+K_b+K_g+K_h) & K_b & K_h & 0 \\ K_b & -(K_c+K_b+K_k+K_j) & 0 & K_k \\ K_h & 0 & (K_e+K_d+K_i+K_h) & K_e \\ 0 & K_k & K_e & -(K_f+K_e+K_1+K_k) \end{bmatrix}\begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix}=\begin{bmatrix} -(100K_a+50K_g) \\ -(100K_c+50K_j) \\ -(100K_d+50K_i) \\ -(100K_f+50K_1) \end{bmatrix}$$

Now, find the value of "K" for all nodes by assumption

$T_1=T_2=T_3=T_4=$ Zero & $K_{ave}=200+0.1\,T_{ave}$ ⇛

$K_a = 200 + 0.1\left(\frac{100+0}{2}\right) = 210$

$K_b = 200 + 0.1\left(\frac{0+0}{2}\right) = 200$

$K_c = 200 + 0.1\left(\frac{100+0}{2}\right) = 210$

$K_d = 200 + 0.1\left(\frac{100+0}{2}\right) = 210$

$K_e = 200 + 0.1\left(\frac{0+0}{2}\right) = 200$

$K_f = 200 + 0.1\left(\frac{100+0}{2}\right) = 210$

$K_g = 200 + 0.1\left(\frac{50+0}{2}\right) = 205$

$K_h = 200 + 0.1\left(\frac{0+0}{2}\right) = 200$

$K_i = 200 + 0.1\left(\frac{50+0}{2}\right) = 205$

$K_j = 200 + 0.1\left(\frac{50+0}{2}\right) = 205$

$K_k = 200 + 0.1\left(\frac{0+0}{2}\right) = 200$

$K_L = 200 + 0.1\left(\frac{50+0}{2}\right) = 205$

So, Sub. in above matrix which leads to:

$$\begin{bmatrix} -815 & 200 & 200 & 0 \\ 200 & -815 & 0 & 200 \\ 200 & 0 & -815 & 200 \\ 0 & 200 & 200 & -815 \end{bmatrix} \cdot \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{bmatrix} = \begin{bmatrix} -31250 \\ -30000 \\ -30250 \\ -31250 \end{bmatrix}$$

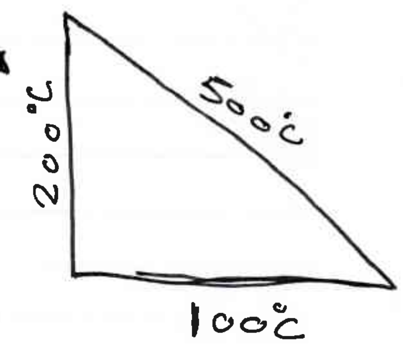
H.W: I) Solve above Ex.9 when $\frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial T}{\partial y}\right) = -xy$

II) Solve $\frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}K\right) + \frac{\partial}{\partial y}\left(\frac{\partial T}{\partial y}K\right) = 0$ when $K = 200 + 0.1T$ & $\Delta x = 1/3$, $\Delta y = 1/4$

for R.B.C $= 50°C$, L.B.C. $= 100°C$, Top B.C. $= 75°C$, & bottom B.C. $= 150°C$

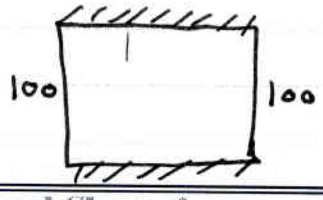
III) Solve $\frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial T}{\partial y}\right) = 0$ & $K = 50 + 0.1T$

$\Delta x = \Delta y = 1/4$



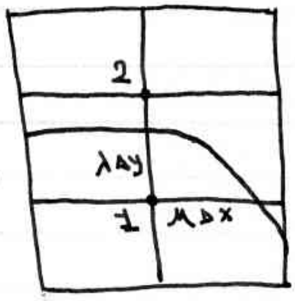
IV) Solve $\frac{\partial}{\partial x}\left(K\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(K\frac{\partial T}{\partial y}\right) = -xy$ &

$K = 50 + 0.1T$ $\Delta x = \Delta y = 1/4$ for



B- Irregular shapes, linear equs., 1, 2, 3 D, steady state:

Ex.16/ Derive an expression for the first & second derivative by finite defference method for curved boundary shown at node (1)



Sol/ By Tayler series forword exp.:-

$$y_{x+\Delta x} = y_x + \Delta x\, y' + \frac{\Delta x^2}{2!} y'' \Big| + \frac{\Delta x^3}{3!} y''' + \cdots$$

cutting, for Irregular shape ⇒

$$y_{x+M\Delta x} = y_x + M\Delta x\, y' + \frac{M^2\Delta x^2}{2!} y'' \quad --- (1)$$

By Tayler series backword exp.:-

$$y_{x-\Delta x} = y_x - \Delta x\, y' + \frac{\Delta x^2}{2!} y'' \Big| - \frac{\Delta x^3}{3!} y''' + \cdots$$

cutting

$$\frac{\Delta x^2}{2!} y'' = y_{x-\Delta x} - y_x + \Delta x\, y' \quad --- (2)$$ sub. in (1) to find (y')

$$y_{x+\Delta x M} = y_x + M\Delta x\, y' + M^2 (y_{x-\Delta x} - y_x + \Delta x\, y')$$

$$= y_x + M\Delta x\, y' + M^2\Delta x\, y' + M^2 y_{x-\Delta x} - M^2 y_x$$

$$= y_x + \Delta x\, y' (M + M^2) + M^2 y_{x-\Delta x} - M^2 y_x$$

$$\left[\; y' = \frac{y_{x+M\Delta x} - y_x - M^2 y_{x-\Delta x} + M^2 y_x}{\Delta x (M + M^2)} \;\right]$$ the first derivative in x-axis

In y-axis, replaced all (M) by (λ)

from Tayler series backword exp. ⇒

$$\Delta x\, y' = y_x - y_{x-\Delta x} + \frac{\Delta x^2}{2!} y'' \quad --- (3)$$ Sub. in (1) to find (y'')

$$y_{x+M\Delta x} = y_x + M\left(y_x - y_{x-\Delta x} + \frac{\Delta x^2}{2!} y''\right) + \frac{M^2\Delta x^2}{2!} y''$$

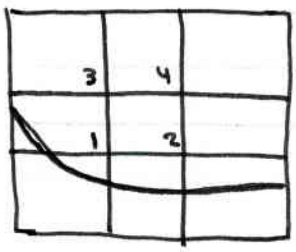
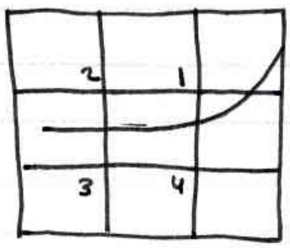
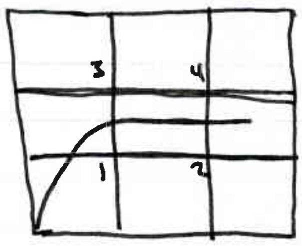
$$= y_x + M y_x - M y_{x-\Delta x} + \frac{M\Delta x^2}{2!} y'' + \frac{M^2\Delta x^2}{2!} y''$$

$$y_{x+\mu\Delta x} = y_x + \Delta x^2 y''(\mu+\mu^2)\frac{1}{2!} + \mu y_x - \mu y_{x-\Delta x}$$

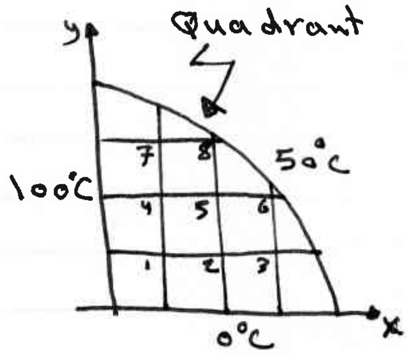
$$\left[\, y'' = \frac{2(y_{x+\mu\Delta x} - y_x - \mu y_{x-\Delta x} + \mu y_{x-\Delta x})}{\Delta x^2(\mu+\mu^2)} \,\right]$$ the second derivative in x-axis

In y-axis, replaced all (μ) by (λ)

H.W: Derive an expression for the first & second derivative by F.D. for this curved shown at node ①

I)  ; II)  ; & III) 

Ex.II: Find the temperature distribution through the flat plate shown. Derive all equation you may need when $\Delta x = \Delta y = 1/4$



Sol: For nodes ①, ②, ④, & ⑤ regular nodes by center exp. ⇒

$$y_{x+\Delta x} = y_x + \Delta x y' + \frac{\Delta x^2}{2!} y'' + \frac{\Delta x^3}{3!} y''' + \dots$$

$$y_{x-\Delta x} = y_x - \Delta x y' + \frac{\Delta x^2}{2!} y'' - \frac{\Delta x^3}{3!} + \dots$$

cutting

Adding

$$y_{x+\Delta x} + y_{x-\Delta x} = 2y_x + \Delta x^2 y''$$

$$y'' = \frac{y_{x+\Delta x} - 2y_x + y_{x-\Delta x}}{\Delta x^2} \Rightarrow \frac{\partial^2 T}{\partial x^2} = \frac{T_{m+1,n} - 2T_{m,n} + T_{m-1,n}}{\Delta x^2}, \quad \frac{\partial^2 T}{\partial y^2} = \frac{T_{m,n+1} - 2T_{m,n} + T_{m,n-1}}{\Delta y^2}$$

Node ①

$$\frac{T_2 - 2T_1 + 100}{\Delta x^2} + \frac{T_4 - 2T_1 + 0}{\Delta y^2} = 0 \quad * \Delta x^2 \Rightarrow$$

$$T_2 + T_4 - 4T_1 = -100 \quad --- ①$$

Node (2)

$$\frac{T_3-2T_2+T_1}{\Delta x^2}+\frac{T_5-2T_2-0}{\Delta y^2}=0 \qquad *\,\Delta x^2 \Rightarrow$$

$$T_1+T_3+T_5-4T_2=0 \qquad ---\ (2)$$

Nodes (4)

$$\frac{T_5-2T_3+100}{\Delta x^2}+\frac{T_7-2T_3+T_1}{\Delta y^2}=0 \qquad *\,\Delta x^2 \Rightarrow$$

$$T_5+T_7+T_1-4T_4=-100 \qquad ---\ (4)$$

Nodes (5)

$$\frac{T_6-2T_5+T_4}{\Delta x^2}+\frac{T_8+2T_5+T_2}{\Delta y^2}=0 \qquad *\,\Delta x^2$$

$$T_6+T_4+T_8+T_2-4T_5=0 \qquad ---\ (5)$$

For nodes (3),(6),(7),&(8) Irregular nodes

$$y_{x+\Delta x}=y_x+\Delta x\,y'+\frac{\Delta x^2}{2!}y'' \qquad \text{Forword exp.}$$

$$y_{x-\Delta x}=y_x-\Delta x\,y'+\frac{\Delta x^2}{2!}y'' \qquad \text{Backword exp.}$$

for nodes irregular forword $\Rightarrow$

$$y_{x+\mu\Delta x}=y_x+\mu\Delta x\,y'+\frac{\mu^2\Delta x^2}{2!}y'' \qquad ---\ (1a)$$

from backword exp. $\Rightarrow y'\Delta x=-y_{x-\Delta x}+y_x+\frac{\mu^2\Delta x^2}{2!}y''$ --- (2a) sub. in

$$y_{x+\mu\Delta x}=y_x+\mu\left(y'+y_{x-\Delta x}+\frac{\Delta x^2}{2!}y''\right)+\frac{\mu^2\Delta x^2}{2}y''$$

$$\left[\,y''=\frac{2(y_{x+\mu\Delta x}-y_x-\mu y_x+\mu y_{x-\Delta x})}{\Delta x^2(\mu+\mu^2)}\,\right]$$

Node (3) $\Delta x=\Delta y=0.25$, $\mu=0.9$ cm (to scale) $*\,0.25 \Rightarrow \mu=0.225$

$$\frac{2(50-T_3-0.225T_3+0.225T_2)}{(0.25)^2(0.225+0.225)^2}+\frac{T_6-2T_3+0}{(0.25)^2}=0 \qquad *(0.25)^2(0.225+0.225^2)$$

$$0.275T_6-0.45T_2-3T_3=-100$$, approach this equ. to $\Rightarrow$

$$0.3T_6 - 0.5T_2 - 3T_3 = -100 \quad ---(3)$$

Node (6) $\Delta x = \Delta y = 0.25$, $M = 0.45$ (to scale) $* 0.25 = 0.113$, $\lambda = 0.6 * 0.25 = 0.15$

$$\frac{2(50 - T_6 - 0.113T_6 + 0.113T_5)}{(0.25)^2(0.113 + 0.113^2)} + \frac{2(50 - T_6 - 0.15T_6 + 0.15(50))}{(0.25)^2(0.15 + 0.15^2)} = 0 \quad * \text{بالمقامين}$$

$$1 - 0.02T_6 - 0.00113T_6 + 0.00113T_5 + 0.016T_6 - 0.0012T_6 + 0.06$$

$$0.00113T_5 - 0.0385T_6 = -1.86 \quad ; \text{approach this equ. to} \Rightarrow$$

$$0.00113T_5 - 0.04T_6 = -2 \quad ---(6)$$

Node (8) $\Delta x = \Delta y = 0.25$, $M = 0.65 * 0.25 = 0.1625$; $\lambda = 0.45 * 0.25 = 0.113$

$$\frac{2(50 - T_8 - 0.16T_8 + 0.16T_7)}{(0.25)^2(0.16 + 0.16^2)} + \frac{2(50 - T_8 - 0.113T_8 + 0.113T_5)}{(0.25)^2(0.113 + 0.113^2)} = 0 \quad * \text{بالمقامين}$$

$$0.0025T_7 + 0.0013T_5 - 0.043T_8 = -1.96 \quad ; \text{approach to} \Rightarrow$$

$$0.003T_7 + 0.0015T_5 - 0.04T_8 = -2 \quad ---(8)$$

Node (7) $\Delta x = \Delta y = 0.25$, $\lambda = 0.225$, $M = 1$

$$\frac{T_8 - 2T_7 + 100}{(0.25)^2} + \frac{2(50 - T_7 - 0.225T_7 + 0.225T_4)}{(0.25)^2(0.225 + 0.225^2)} = 0 \quad * (0.25)^2(0.225 + 0.225^2)$$

$$0.275T_8 - 0.551T_7 + 27.5 + 100 - 2T_7 - 0.45T_7 + 0.45T_4 = 0$$

$$0.75T_8 + 0.45T_4 - 3T_7 = -127.5 \quad \text{approach to} \Rightarrow$$

$$0.3T_8 + 0.5T_4 - 3T_7 = -128 \quad ---(7)$$

Now, we will write matrix for all 8th equs. $\Rightarrow$

$$\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0.5 & -3 & 0 & 0 & 0.3 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0.001 & -0.04 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 & 0 & -3 & 0.3 \\ 0 & 0 & 0 & 0 & 0.0015 & 0 & 0.003 & -0.04 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ T_6 \\ T_7 \\ T_8 \end{bmatrix} = \begin{bmatrix} -100 \\ 0 \\ -100 \\ -100 \\ 0 \\ -2 \\ -128 \\ -2 \end{bmatrix}$$

Sol. H.W, I) By Tayler series forward exp. :-

$$y_{x+\Delta x} = y_x + \Delta x\, y' + \frac{\Delta x^2}{2!} y'' \Big|_{\text{cutting}} + \frac{\Delta x^3}{3!} y''' + \cdots \quad , \text{for irregular shape}$$

$$\frac{\Delta x^2}{2!} y'' = y_{x+\Delta x} - y_x - \Delta x\, y' \quad \text{---} \ (1)$$

$$y_{x-\Delta x} = y_x - \Delta x\, y' + \frac{\Delta x^2}{2!} y'' \Big|_{\text{cutting}} + \frac{\Delta x^3}{3!} y''' + \cdots \quad , \text{for irregular shape}$$

$$y_{x-\mu\Delta x} = y_x - \mu \Delta x\, y' + \frac{\mu^2 \Delta x^2}{2!} y'' \quad \text{---} \ (2) \quad \text{sub. (1) in (2)} \Rightarrow \text{To find } (y')$$

$$y_{x-\mu\Delta x} = y_x - \mu \Delta x\, y' + \mu^2 (y_{x+\Delta x} - y_x - \Delta x\, y')$$

$$y_{x-\mu\Delta x} = y_x - \mu \Delta x\, y' + \mu^2 y_{x+\Delta x} - \mu^2 y_x - \mu^2 \Delta x\, y'$$

$$-y_{x-\mu\Delta x} + y_x + \mu^2 (y_{x+\Delta x} - y_x) = y' (\mu + \mu^2) \Delta x$$

$$\left[y' = \frac{y_x + \mu^2 (y_{x+\Delta x} - y_x) - y_{x-\mu\Delta x}}{\Delta x (\mu + \mu^2)} \right]$$

the first derivative in x-axis.
In y-axis, replaced all (μ) by (λ).

from Tayler series forword exp. $\Rightarrow$

$$\Delta x\, y' = y_{x+\Delta x} - y_x - \frac{\Delta x^2}{2!} y'' \quad \text{---} \ (3) \quad \text{sub. in (2) to find } (y'') \Rightarrow$$

$$y_{x-\mu\Delta x} = y_x - \mu \left(y_{x+\Delta x} - y_x - \frac{\Delta x^2}{2!} y'' \right) + \frac{\Delta x^2}{2!} y''$$

$$y_{x-\mu\Delta x} = y_x - \mu y_{x+\Delta x} + \mu y_x + \mu \Delta x^2/2 \; y'' + \frac{\Delta x^2}{2} y''$$

$$= y_x + \mu(y_x - y_{x+\Delta x}) + \frac{\Delta x^2}{2} y'' (\mu + \mu^2)$$

$$\left[y'' = \frac{2\left(y_{x-\mu\Delta x} - \mu(y_x - y_{x+\Delta x}) - y_x\right)}{\Delta x^2 (\mu + \mu^2)} \right]$$

the second derivative in x-axis.
In y-axis, replaced all (μ) by (λ)