Ch.6: Introduction to Finite Elements Methods

One dimension One Element, Two, Three & Four Elements.

Let's consider this element shown as fig.(1):

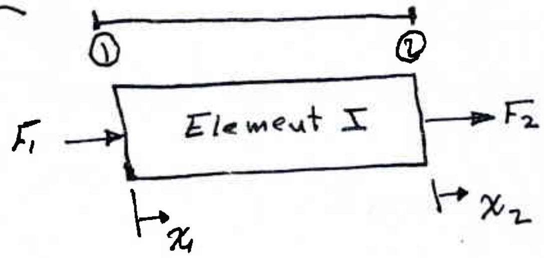


fig.(1)
[1D & 1E]

F_1 & F_2 = Forces on elements.

x_1 & x_2 = Displacement of element.

$\Delta x = \delta$ =

$$\delta = \frac{F \cdot L}{E \cdot A} \quad \text{-----} (1) \rightarrow \frac{F}{\delta} = \frac{E \cdot A}{L} \rightarrow K = \frac{E \cdot A}{L} \quad \therefore F = K \cdot \delta \quad \text{---} (2)$$

E = Modulus of elasticity.

L = Length of element.

A = Cross section area of element.

So, The general equation of element:

$$F = K \cdot \delta = K \cdot \Delta x$$

مصفوفة الطاقات

$$F_1 = K_1(x_1 - x_2) = K_1x_1 - K_1x_2$$

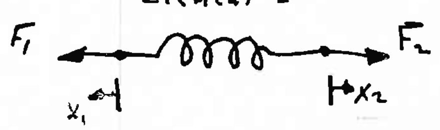
$$F_2 = K_1(x_2 - x_1) = K_1x_1 - K_1x_2$$

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} K_1 & -K_1 \\ -K_1 & K \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

مصفوفة الثوابت مصفوفة المتغيرات

* This is for one element,

Or in a series of springs:



[one dim. & one element]

$F = K \cdot \Delta x \quad ; \Delta x =$

$$-F_1 = K_1(x_1 - x_2) = K_1x_1 - K_1x_2$$

$$F_2 = K_1(x_2 - x_1) = K_1x_2 - K_1x_1$$

$$\begin{bmatrix} -F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} K_1 & -K_1 \\ -K & K_1 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$F_3 = -K_2 \cdot X_2 + K_2 X_3$

$F_3 = -1.87 \times 10^6 (566.5 \times 10^{-6}) + \text{Zero} \rightarrow \boxed{F_3 = -1059.5\ N}$ Ans.②

الاشارة السالبة تعني تغير الاتجاه

* لأن لا نحتاج (F_1)، نلغي المعادلة الثالثة من المصفوفة العامة:

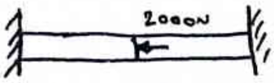
$$\begin{bmatrix} F_1 \\ -2000 \end{bmatrix} = \begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1+K_2 \end{bmatrix} \begin{bmatrix} 0 \\ X_2 \end{bmatrix}$$

$F_1 = K_1 (X_1) - K_1 (X_2)$

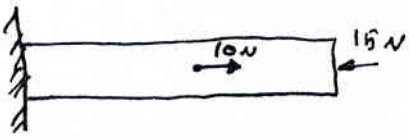
الاشارة السالبة تعني تغير الاتجاه

$F_1 = \text{Zero} - 1.66 \times 10^6 (566.5 \times 10^{-6}) \rightarrow \boxed{F_1 = -940.4\ N}$ Ans.③

H.W. Re-solve the above Ex. 1 when $A_1 = A_2 = 2\,m^2$, $E_1 = E_2 = 10 \times 10^6$ & $L_1 = L_2 = 12\,m$. [1D & 2E]



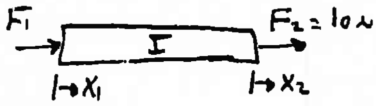
Ex. 2: Find the displacement, force & stress in the system as shwon in fig.
$A = 10 \times 10^{-4}\,m$, $K_1 = K_2 = 1$



Sol: The general Equs. for this system;

$F = K \cdot \delta = K \cdot \Delta X$

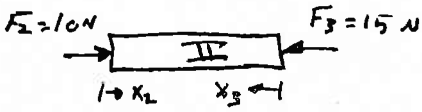
Element (I) $F_1 = K_1 (X_1 - X_2)$



$F_2 = K_1 (X_2 - X_1)$

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1 \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

Element (II) $F_2 = K_2 (X_2 - X_3)$



$-F_3 = K_2 (X_3 - X_2)$

$$\begin{bmatrix} F_2 \\ -F_3 \end{bmatrix} = \begin{bmatrix} K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

∴ The general Matrix for this system:

$$\begin{bmatrix} F_1 \\ F_2 \\ -F_3 \end{bmatrix} = \begin{bmatrix} K_1 & -K_1 & 0 \\ -K_1 & K_1+K_2 & -K_2 \\ 0 & -K_2 & K_2 \end{bmatrix} \times \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix}$$, Now applied B.C. on this G.M.

(from Element II)

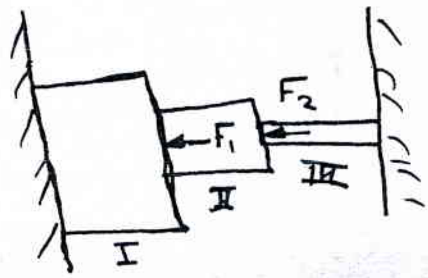
∵ $\frac{E.A.}{L} = K = 1$ ∴ $K_1 = K_2 =$ for same material

Ex. 3: Solve this system by Finite elements to find Reaction & displacements when:

Elements / Properties	I	II	III
A	3	2	1
E	$10*10^6$	$20*10^6$	$30*10^6$
L	20	20	20

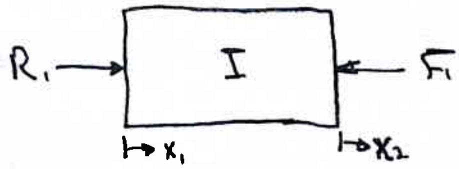
& $F_1 = 4000\ N$

$F_2 = 6000\ N$



Sol. General Equs. for this system:

$F = K \cdot \delta = K \cdot \Delta X$



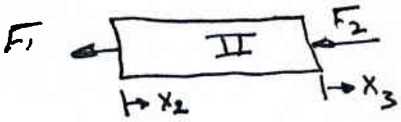
Element (I) $R_1 = K_1(X_1 - X_2)$

$-F_2 = K_1(X_2 - X_1)$

$$\begin{bmatrix} R_1 \\ -F_1 \end{bmatrix} = K_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \rightarrow \begin{bmatrix} R_1 \\ -F_1 \end{bmatrix} = \begin{bmatrix} K_1 & -K_1 \\ -K & K_1 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

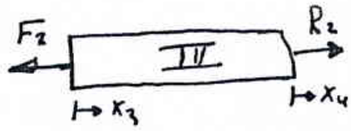
Element (II) $-F_1 = (X_2 - X_3) K_2$

$-F_2 = (X_3 - X_2) K_2$



$$\begin{bmatrix} -F_1 \\ -F_2 \end{bmatrix} = K_2 \begin{bmatrix} 1 & -1 \\ & 1 \end{bmatrix} * \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \rightarrow$$

$$\begin{bmatrix} -F_1 \\ -F_2 \end{bmatrix} = \begin{bmatrix} K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} * \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}$$



Element (III) $-F_2 = K_3(x_3 - x_4)$

$R_2 = K_3(x_4 - x_3)$

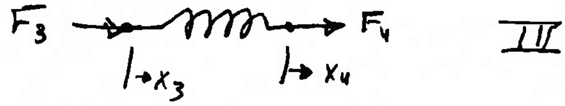
$$\begin{bmatrix} -F_2 \\ R_2 \end{bmatrix} = K_3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} * \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \rightarrow \begin{bmatrix} -F_2 \\ R_2 \end{bmatrix} = \begin{bmatrix} K_3 & -K_3 \\ -K_3 & K_3 \end{bmatrix} * \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$$

$$\begin{bmatrix} R_1 \\ F_1 \\ F_2 \\ R_2 \end{bmatrix} = \begin{bmatrix} K_1 & -K_1 & 0 & 0 \\ -K_1 & K_1+K_2 & -K_2 & 0 \\ 0 & -K_2 & K_2+K_3 & -K_3 \\ 0 & 0 & -K_3 & K_3 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$ This is General Matrix

B.C. $x_1 = 0$, $x_4 = 0$, $F_1 = 4000$, $F_2 = 6000$

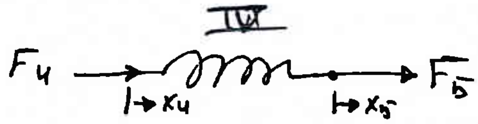
$K_1 = \frac{E_1 A_1}{L_1} = \frac{10*10^6*3}{20} \rightarrow K_1 = 15*10^5$, $K_2 = \frac{E_2 A_2}{L_2} = \frac{20*10^6*2}{20} \rightarrow K_2 = 20*10^5$

Element III) $F_3 = K_3(X_3 - X_4)$
$F_4 = K_3(X_4 - X_3)$



$$\begin{bmatrix} F_3 \\ F_4 \end{bmatrix} = K_3 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} X_3 \\ X_4 \end{bmatrix} \rightarrow \begin{bmatrix} F_3 \\ F_4 \end{bmatrix} = \begin{bmatrix} K_3 & -K_3 \\ -K_3 & K_3 \end{bmatrix} \cdot \begin{bmatrix} X_3 \\ X_4 \end{bmatrix}$$

Element IV) $F_4 = K_4(X_4 - X_5)$
$F_5 = K_4(X_5 - X_4)$



$$\begin{bmatrix} F_4 \\ F_5 \end{bmatrix} = K_4 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} X_4 \\ X_5 \end{bmatrix} \rightarrow \begin{bmatrix} F_4 \\ F_5 \end{bmatrix} = \begin{bmatrix} K_4 & -K_4 \\ -K_4 & K_4 \end{bmatrix} \cdot \begin{bmatrix} X_4 \\ X_5 \end{bmatrix}$$

Now, The general equ. for this system is.

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{bmatrix} = \begin{bmatrix} K_1 & -K_1 & 0 & 0 & 0 \\ -K_1 & K_1+K_2 & -K_2 & 0 & 0 \\ 0 & -K_2 & K_2+K_3 & -K_3 & 0 \\ 0 & 0 & -K_3 & K_3+K_4 & -K_4 \\ 0 & 0 & 0 & -K_4 & K_4 \end{bmatrix} * \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{bmatrix}$$

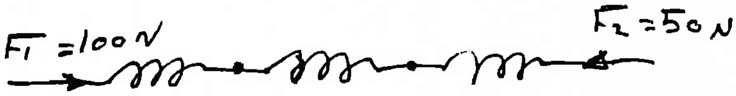
E.I   E.II   E.III   E.IV

B.C. $K_1 = K_3 = 1$, $K_2 = K_4 = 2$, $X_5 = \text{zero}$, $F_1 = -10\,N$

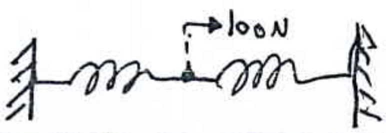
$$\begin{bmatrix} -10 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 3 & -2 & 0 & 0 \\ 0 & -2 & 3 & -1 & 0 \\ 0 & 0 & -1 & 3 & -2 \\ 0 & 0 & 0 & -2 & 2 \end{bmatrix} * \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ 0 \end{bmatrix}$$

H.W: Solve by Finite elements for the series of spring when; $K_1 = K_2 = 2$ & $K_3 = K_4 = 4$ with Tow forces; $F_1 = 100\,N$, $F_2 = 100\,N$

H.W: By finite elements; solve this system of spring when;
$K_1 = 1$, $K_2 = 2$, $K_3 = 3$,



H.W: By [F.E.] solve this system of spring when; $K_1 = 2$, $K_2 = 2K_1$

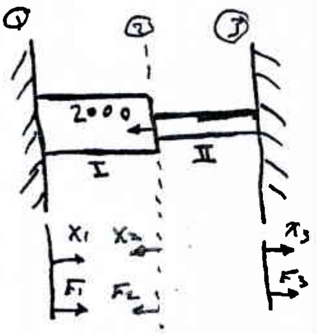


Ex.1: Using Finite elements to find the forces & displacemen of this system

for; $A_1 = 2\ m^2$, $E_1 = 10*10^6\ N$, $L_1 = 12\ m$

$A_2 = 1\ m^2$, $E_2 = 30*10^6\ N$, $L_2 = 16\ m$

[1D & 2E]

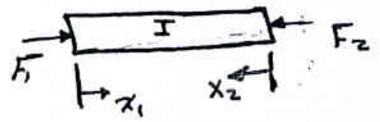


Sol: General Equ. for this system:

$$F = K.\delta = K.\Delta x$$

Element (I) $F_1 = K_1(x_1 - x_2) = K_1 x_1 - K_1 x_2$

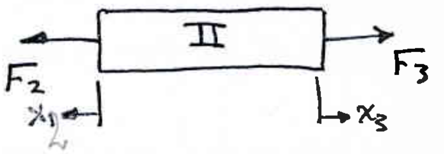
$F_2 = K_1(x_2 - x_1) = K_1 x_2 - K_1 x_1$



$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Element (II) $F_2 = K_2(x_2 - x_3) = K_2 x_2 - K_2 x_3$

$F_3 = K_2(x_3 - x_2) = K_2 x_3 - K_2 x_2$



$$\begin{bmatrix} F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} * \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}$$

$$\begin{matrix} K_1 & -K_1 & 0 \\ -K_1 & K_1+K_2 & -K_2 \\ 0 & -K_2 & K_2 \end{matrix}$$

There are three equs. F_1, F_2, & F_3.

The general Matrix for this system;

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} K_1 & -K_1 & 0 \\ -K_1 & K_1+K_2 & -K_2 \\ 0 & -K_2 & K_2 \end{bmatrix} * \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

, Now applied **B.C** on the **G.M.**

from Element I — from Element II

$$\begin{bmatrix} F_1 \\ -2000 \\ F_3 \end{bmatrix} = \begin{bmatrix} K_1 & -K_1 & 0 \\ -K_1 & K_1+K_2 & -K_2 \\ 0 & -K_2 & K_2 \end{bmatrix} * \begin{bmatrix} 0 \\ x_2 \\ 0 \end{bmatrix},$$

$$K_1 = \frac{E_1 \cdot A_1}{L_1} = \frac{10*10^6 * 2}{12} \Rightarrow K_1 = 1.66*10^6$$

$$K_2 = \frac{E_2 \cdot A_2}{L_2} = \frac{30*10^6 * 1}{16} \Rightarrow K_2 = 1.87*10^6$$

الآن نلغي المعادلة الأولى من المصفوفة أعلاه:

$$\begin{bmatrix} -2000 \\ F_3 \end{bmatrix} = \begin{bmatrix} K_1+K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{bmatrix} x_2 \\ 0 \end{bmatrix}$$; Solve by Chapter 2 or

$$-2000 = (K_1+K_2)x_2 - K_2 x_3$$

$$-2000 = 3.53*10^6 (x_2) - 1.87*10^6 (\cancel{x_3})^{0} \Rightarrow \boxed{x_2 = 566.5*10^{-6}}$$ **Ans(1)**

$$\begin{bmatrix} F_1 \\ 10 \\ -15 \end{bmatrix} = \begin{bmatrix} K & -K & 0 \\ -K & 2K & -K \\ 0 & -K & K \end{bmatrix} * \begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix}$$

* الان نلغي المعادلات الاولى من المصفوفة العامة اعلاه :

$$\begin{bmatrix} 10 \\ -15 \end{bmatrix} = \begin{bmatrix} 2K & -K \\ -K & K \end{bmatrix} * \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}$$; solve by Chapter $\underline{\underline{2}}$ or

$10 = 2K\,x_2 - K\,x_3 \rightarrow K\,x_3 = 2K\,x_2 - 10$ --- ①

$-15 = -K\,x_2 + K\,x_3$ --- ②

Now, Sub. ① in ② $\Rightarrow -15 = -K\,x_2 + 2K\,x_2 - 10$

$$-5 = K\,x_2$$

$$\boxed{x_2 = -\frac{5}{K}}$$ sub. in ① $\rightarrow$

$$\boxed{x_3 = -\frac{20}{K}}$$ <u>Ans.</u> ①

* الان نلغي المعادلات الثالثة من المصفوفة العامة : لاستخراج (F_1)

$$\begin{bmatrix} F_1 \\ 10 \end{bmatrix} = \begin{bmatrix} K & -K \\ -K & 2K \end{bmatrix} * \begin{bmatrix} 0 \\ x_2 \end{bmatrix}$$; solve this Matrix (<u>H.W.</u>) or

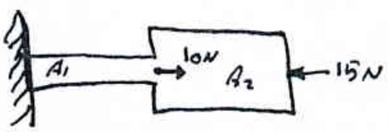
$F_1 = K(0) - K\,x_2 \rightarrow F_1 = -K * \frac{-5}{K} \rightarrow \boxed{F_1 = 5N}$ <u>Ans.</u> ②

$\sigma_1 = \frac{F_1}{A} = \frac{5}{10 \times 10^{-4}} \rightarrow \sigma_1 = 5000\ N/m^2$ <u>Ans.</u> ③

$\sigma_2 = \frac{F_2}{A} = \frac{10}{10 \times 10^{-4}} \rightarrow \sigma_2 = 10\,000\ N/m^2$ <u>Ans.</u> ④

$\sigma_3 = \frac{F_3}{A} = \frac{15}{10 \times 10^{-4}} \rightarrow \sigma_3 = 15\,000\ N/m^2$ <u>Ans</u> ⑤

<u>H.W.</u> Solve the above Ex.$\underline{2}$ when $A_1 = 10 \times 10^{-4}\ m^2$, $E_1 = 10 \times 10^6\ N/m^2$ &
$A_2 = 20 \times 10^{-4}\ m^2$, $E_2 = 20 \times 10^6\ N/m^2$
$L_1 = L_2 = 10\ m$
by using (F.E.).

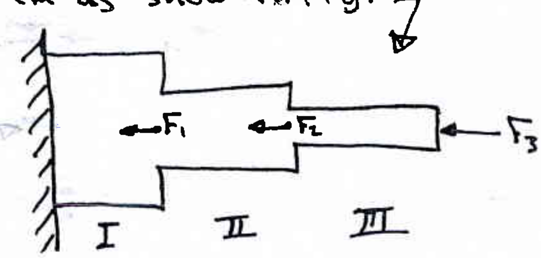


Note: Ex.1 & Ex.2 No. elements are 2.

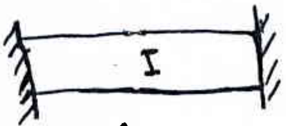
$$K_3 = \frac{E_3 A_3}{L_3} = \frac{30 * 10^6 * 1}{20} \rightarrow K_3 = 15 * 10^5$$

$$\begin{bmatrix} R_1 \\ -4000 \\ -6000 \\ R_2 \end{bmatrix} = \begin{bmatrix} 15*10^5 & -15*10^5 & 0 & 0 \\ -15*10^5 & 35*10^5 & -20*10^5 & 0 \\ 0 & -20*10^5 & 35*10^5 & -15*10^5 \\ 0 & 0 & -15*10^5 & 15*10^5 \end{bmatrix} * \begin{bmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{bmatrix}$$

H.W.) Solve The Ex.3 when $\frac{E.A}{L} = 1$ & the system as show in fig.
$F_1 = 1000N$, $F_2 = 2000N$
by Finite elements.



H.W.) Solve by (F.E.) for this system as show in figs. when
$A = 3\,m^2$, $E = 30*10^6$, $L = 10m$



fig(3)

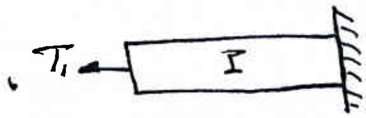


fig.(2)

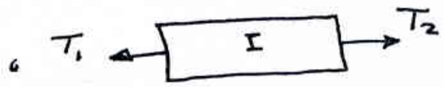


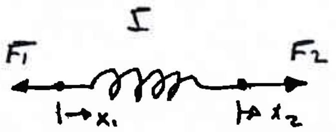
fig.(1)

Ex. 4: Solve the series of spring by Finite elements when $K_1 = 1 = K_3$, $K_2 = 2 = K_4$.
$F_1 = 10\,N$

Sol.) The general equ. is. $F = K \cdot \Delta X$

Element I) $-F_1 = K_1(X_1 - X_2)$

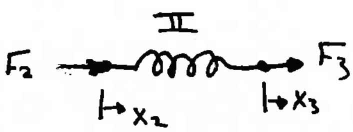
$F_2 = K_1(X_2 - X_1)$



$$\begin{bmatrix} -F_1 \\ F_2 \end{bmatrix} = K_1 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \rightarrow \begin{bmatrix} -F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} K_1 & -K_1 \\ -K_1 & K_1 \end{bmatrix} \cdot \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

Element II) $F_2 = K_2(X_2 - X_3)$

$F_3 = K_2(X_3 - X_2)$



$$\begin{bmatrix} F_2 \\ F_3 \end{bmatrix} = K_2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} X_2 \\ X_3 \end{bmatrix} \rightarrow \begin{bmatrix} F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \cdot \begin{bmatrix} X_2 \\ X_3 \end{bmatrix}$$