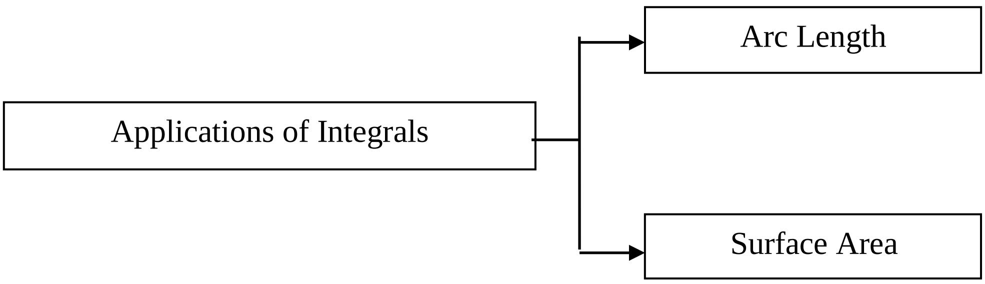
By

Ms.C. Yasir R. Al-hamdany

Lecture – Six

Applications of Integrals

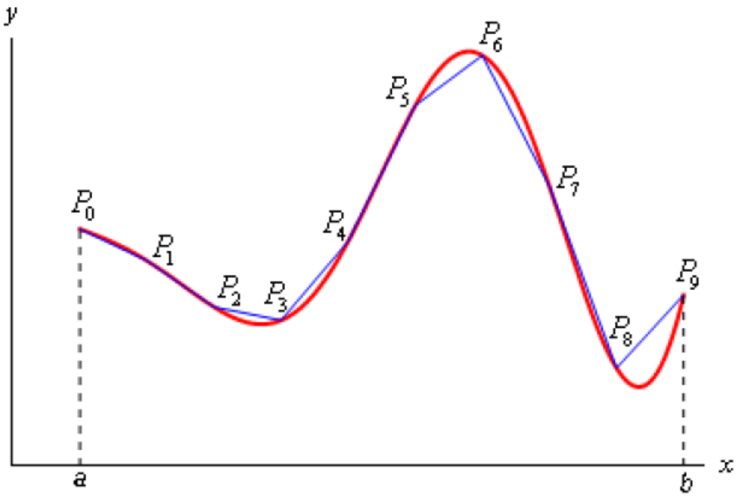


1- ***Applications of Integrals.***

A- ***Arc Length.***

We want to determine the length of the continuous function $y = f(x)$ on the interval $[a,b]$. We'll also need to assume that the derivative is continuous on $[a,b]$.

Initially we'll need to estimate the length of the curve. We'll do this by dividing the interval up into n equal subintervals each of width Δx and we'll denote the point on the curve at each point by P_i. We can then approximate the curve by a series of straight lines connecting the points. Here is a sketch of this situation for $n = 9$.



Now denote the length of each of these line segments by $|P_{i-1}\ P_i|$ and the length of the curve will then be approximately,

$$L \approx \sum_{i=1}^{n} |P_{i-1}\ P_i|$$

and we can get the exact length by taking n larger and larger. In other words, the exact length will be,

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}\ P_i|$$

Now, let's get a better grasp on the length of each of these line segments. First, on each segment let's define $\Delta y_i = y_i - y_{i-1} = f(x_i) - f(x_{i-1})$. We can then compute directly the length of the line segments as follows.

$$|P_{i-1}\ P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{\Delta x^2 + \Delta y_i^2}$$

By the Mean Value Theorem we know that on the interval $[x_{i-1}, x_i]$ there is a point x_i^* so that,

$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})$$
$$\Delta y_i = f'(x_i^*)\,\Delta x$$

Therefore, the length can now be written as,

$$\begin{aligned} |P_{i-1}\ P_i| &= \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} \\ &= \sqrt{\Delta x^2 + \left[f'(x_i^*)\right]^2 \Delta x^2} \\ &= \sqrt{1 + \left[f'(x_i^*)\right]^2}\ \ \Delta x \end{aligned}$$

$$f(x_i) - f(x_{i-1}) = f'(x_i^*)(x_i - x_{i-1})$$
$$\Delta y_i = f'(x_i^*)\,\Delta x$$

Therefore, the length can now be written as,

$$\begin{aligned} |P_{i-1}\ P_i| &= \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} \\ &= \sqrt{\Delta x^2 + \left[f'(x_i^*)\right]^2 \Delta x^2} \\ &= \sqrt{1 + \left[f'(x_i^*)\right]^2}\ \ \Delta x \end{aligned}$$

The exact length of the curve is then,

$$\begin{aligned} L &= \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}\ P_i| \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left[f'(x_i^*)\right]^2}\ \ \Delta x \end{aligned}$$

The exact length of the curve is then,

$$\begin{aligned} L &= \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}\ P_i| \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1 + \left[f'(x_i^*)\right]^2}\ \ \Delta x \end{aligned}$$

However, using the definition of the definite integral, this is nothing more than,

$$L = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\, dx$$

A slightly more convenient notation (in my opinion anyway) is the following.

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

In a similar fashion we can also derive a formula for $x = h(y)$ on $[c,d]$. This formula is,

$$L = \int_c^d \sqrt{1 + [h'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Arc Length Formula(s)

$$L = \int ds$$

where,

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{if } y = f(x),\ a \le x \le b$$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \qquad \text{if } x = h(y),\ c \le y \le d$$

Example 1 Determine the length of $y = \ln(\sec x)$ between $0 \le x \le \frac{\pi}{4}$.

Solution

In this case we'll need to use the first ds since the function is in the form $y = f(x)$. So, let's get the derivative out of the way.

$$\frac{dy}{dx} = \frac{\sec x \tan x}{\sec x} = \tan x \qquad\qquad \left(\frac{dy}{dx}\right)^2 = \tan^2 x$$

Let's also get the root out of the way since there is often simplification that can be done and there's no reason to do that inside the integral.

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \tan^2 x} = \sqrt{\sec^2 x} = |\sec x| = \sec x$$

Note that we could drop the absolute value bars here since secant is positive in the range given.

The arc length is then,

$$\begin{aligned} L &= \int_0^{\frac{\pi}{4}} \sec x\, dx \\ &= \ln|\sec x + \tan x|\Big|_0^{\frac{\pi}{4}} \\ &= \ln\left(\sqrt{2} + 1\right) \end{aligned}$$

Example 2 Determine the length of $x = \frac{2}{3}\left(y-1\right)^{\frac{3}{2}}$ between $1 \le y \le 4$.

Solution

Let's compute the derivative and the root.

$$\frac{dx}{dy} = \left(y-1\right)^{\frac{1}{2}} \qquad \Rightarrow \qquad \sqrt{1+\left(\frac{dx}{dy}\right)^2} = \sqrt{1+y-1} = \sqrt{y}$$

As you can see keeping the function in the form $x = h\left(y\right)$ is going to lead to a very easy integral. To see what would happen if we tried to work with the function in the form $y = f\left(x\right)$ see the next example.

Let's get the length.

$$\begin{aligned} L &= \int_1^4 \sqrt{y}\, dy \\ &= \left.\frac{2}{3}y^{\frac{3}{2}}\right|_1^4 \\ &= \frac{14}{3} \end{aligned}$$

Example 3 Redo the previous example using the function in the form $y = f\left(x\right)$ instead.

Solution

In this case the function and its derivative would be,

$$y = \left(\frac{3x}{2}\right)^{\frac{2}{3}} + 1 \qquad\qquad \frac{dy}{dx} = \left(\frac{3x}{2}\right)^{-\frac{1}{3}}$$

The root in the arc length formula would then be.

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{1}{\left(\frac{3x}{2}\right)^{\frac{2}{3}}}} = \sqrt{\frac{\left(\frac{3x}{2}\right)^{\frac{2}{3}}+1}{\left(\frac{3x}{2}\right)^{\frac{2}{3}}}} = \frac{\sqrt{\left(\frac{3x}{2}\right)^{\frac{2}{3}}+1}}{\left(\frac{3x}{2}\right)^{\frac{1}{3}}}$$

All the simplification work above was just to put the root into a form that will allow us to do the integral.

Now, before we write down the integral we'll also need to determine the limits. This particular ds requires x limits of integration and we've got y limits. They are easy enough to get however. Since we know x as a function of y all we need to do is plug in the original y limits of integration and get the x limits of integration. Doing this gives,

$$0 \le x \le \frac{2}{3}\left(3\right)^{\frac{3}{2}}$$

Not easy limits to deal with, but there they are.

Let's now write down the integral that will give the length.

$$L = \int_0^{\frac{2}{3}(3)^{\frac{3}{2}}} \frac{\sqrt{\left(\frac{3x}{2}\right)^{\frac{2}{3}} + 1}}{\left(\frac{3x}{2}\right)^{\frac{1}{3}}}\, dx$$

That's a really unpleasant looking integral. It can be evaluated however using the following substitution.

$$u = \left(\frac{3x}{2}\right)^{\frac{2}{3}} + 1 \qquad\qquad du = \left(\frac{3x}{2}\right)^{-\frac{1}{3}} dx$$

$$x = 0 \qquad \Rightarrow \qquad u = 1$$

$$x = \frac{2}{3}(3)^{\frac{3}{2}} \qquad \Rightarrow \qquad u = 4$$

Using this substitution the integral becomes,

$$\begin{aligned} L &= \int_1^4 \sqrt{u}\, du \\ &= \frac{2}{3}u^{\frac{3}{2}}\Big|_1^4 \\ &= \frac{14}{3} \end{aligned}$$

So, we got the same answer as in the previous example. Although that shouldn't really be all that surprising since we were dealing with the same curve.

Example 4 Determine the length of $x = \frac{1}{2}y^2$ for $0 \le x \le \frac{1}{2}$. Assume that y is positive.

Solution

We'll use the second ds for this one as the function is already in the correct form for that one. Also, the other ds would again lead to a particularly difficult integral. The derivative and root will then be,

$$\frac{dx}{dy} = y \qquad \Rightarrow \qquad \sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + y^2}$$

Before writing down the length notice that we were given x limits and we will need y limits for this ds. With the assumption that y is positive these are easy enough to get. All we need to do is plug x into our equation and solve for y. Doing this gives,

$$0 \le y \le 1$$

The integral for the arc length is then,

$$L = \int_0^1 \sqrt{1 + y^2}\, dy$$

This integral will require the following trig substitution.

$$y = \tan\theta \qquad\qquad dy = \sec^2\theta\, d\theta$$

$$y = 0 \quad \Rightarrow \quad 0 = \tan\theta \quad \Rightarrow \quad \theta = 0$$

$$y = 1 \quad \Rightarrow \quad 1 = \tan\theta \quad \Rightarrow \quad \theta = \frac{\pi}{4}$$

$$\sqrt{1+y^2} = \sqrt{1+\tan^2\theta} = \sqrt{\sec^2\theta} = |\sec\theta| = \sec\theta$$

The length is then,

$$L = \int_0^{\frac{\pi}{4}} \sec^3\theta \, d\theta$$

$$= \frac{1}{2}\left(\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\right)\Big|_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left(\sqrt{2} + \ln\left(1+\sqrt{2}\right)\right)$$

B- ***Surface Area.***

The surface area of a frustum is given by,

$$A = 2\pi r l$$

where,

$$r = \frac{1}{2}(r_1 + r_2) \qquad r_1 = \text{radius of right end}$$

$$r_2 = \text{radius of left end}$$

and l is the length of the slant of the frustum.

For the frustum on the interval $[x_{i-1}, x_i]$ we have,

$$r_1 = f(x_i)$$

$$r_2 = f(x_{i-1})$$

$$l = |P_{i-1}\,P_i| \quad (\text{length of the line segment connecting } P_i \text{ and } P_{i-1})$$

We know from the previous section that,

$$|P_{i-1}\,P_i| = \sqrt{1+\left[f'(x_i^*)\right]^2}\,\Delta x \quad \text{where } x_i^* \text{ is some point in } [x_{i-1}, x_i]$$

Before writing down the formula for the surface area we are going to assume that Δx is “small” and since $f(x)$ is continuous we can then assume that,

$$f(x_i) \approx f(x_i^*) \qquad \text{and} \qquad f(x_{i-1}) \approx f(x_i^*)$$

So, the surface area of the frustum on the interval $[x_{i-1}, x_i]$ is approximately,

$$A_i = 2\pi\left(\frac{f(x_i)+f(x_{i-1})}{2}\right)\left|P_{i-1}\;P_i\right|$$
$$\approx 2\pi f(x_i^*)\sqrt{1+\left[f'(x_i^*)\right]^2}\;\;\Delta x$$

The surface area of the whole solid is then approximately,

$$S \approx \sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1+\left[f'(x_i^*)\right]^2}\;\;\Delta x$$

and we can get the exact surface area by taking the limit as n goes to infinity.

$$S = \lim_{n\to\infty}\sum_{i=1}^{n} 2\pi f(x_i^*)\sqrt{1+\left[f'(x_i^*)\right]^2}\;\;\Delta x$$
$$= \int_a^b 2\pi f(x)\sqrt{1+\left[f'(x)\right]^2}\,dx$$

If we wanted to we could also derive a similar formula for rotating $x = h(y)$ on $[c,d]$ about the y-axis. This would give the following formula.

$$S = \int_c^d 2\pi\, h(y)\sqrt{1+\left[h'(y)\right]^2}\,dy$$

Surface Area Formulas

$$S = \int 2\pi y\,ds \qquad \text{rotation about } x-\text{axis}$$
$$S = \int 2\pi x\,ds \qquad \text{rotation about } y-\text{axis}$$

where,

$$ds = \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad \text{if } y = f(x),\; a \le x \le b$$
$$ds = \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy \qquad \text{if } x = h(y),\; c \le y \le d$$

Example 1 Determine the surface area of the solid obtained by rotating $y = \sqrt{9-x^2}$, $-2 \le x \le 2$ about the x-axis.

Solution

The formula that we'll be using here is,

$$S = \int 2\pi y\,ds$$

Let's first get the derivative and the root taken care of.

$$\frac{dy}{dx} = \frac{1}{2}\left(9-x^2\right)^{-\frac{1}{2}}(-2x) = -\frac{x}{\left(9-x^2\right)^{\frac{1}{2}}}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{x^2}{9-x^2}} = \sqrt{\frac{9}{9-x^2}} = \frac{3}{\sqrt{9-x^2}}$$

Here's the integral for the surface area,

$$S = \int_{-2}^{2} 2\pi y \frac{3}{\sqrt{9-x^2}}\,dx$$

There is a problem however. The dx means that we shouldn't have any y's in the integral. So, before evaluating the integral we'll need to substitute in for y as well.

The surface area is then,

$$\begin{aligned} S &= \int_{-2}^{2} 2\pi\sqrt{9-x^2}\,\frac{3}{\sqrt{9-x^2}}\,dx \\ &= \int_{-2}^{2} 6\pi\,dx \\ &= 24\pi \end{aligned}$$

Example 2 Determine the surface area of the solid obtained by rotating $y = \sqrt[3]{x}$, $1 \le y \le 2$ about the y-axis. Use both ds's to compute the surface area.

Solution
Note that we've been given the function set up for the first ds and limits that work for the second ds.

Solution 1
This solution will use the first ds listed above. We'll start with the derivative and root.

$$\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}}$$

$$\sqrt{1+\left(\frac{dy}{dx}\right)^2} = \sqrt{1+\frac{1}{9x^{\frac{4}{3}}}} = \sqrt{\frac{9x^{\frac{4}{3}}+1}{9x^{\frac{4}{3}}}} = \frac{\sqrt{9x^{\frac{4}{3}}+1}}{3x^{\frac{2}{3}}}$$

We'll also need to get new limits. That isn't too bad however. All we need to do is plug in the given y's into our equation and solve to get that the range of x's is $1 \le x \le 8$. The integral for the surface area is then,

$$\begin{aligned} S &= \int_{1}^{8} 2\pi x \frac{\sqrt{9x^{\frac{4}{3}}+1}}{3x^{\frac{2}{3}}}\,dx \\ &= \frac{2\pi}{3}\int_{1}^{8} x^{\frac{1}{3}}\sqrt{9x^{\frac{4}{3}}+1}\,dx \end{aligned}$$

Using the substitution

$$u = 9x^{\frac{4}{3}} + 1 \qquad\qquad du = 12x^{\frac{1}{3}}\,dx$$

the integral becomes,

$$S = \frac{\pi}{18}\int_{10}^{145} \sqrt{u}\,du$$

$$= \frac{\pi}{27} u^{\frac{3}{2}} \Big|_{10}^{145}$$

$$= \frac{\pi}{27}\left(145^{\frac{3}{2}} - 10^{\frac{3}{2}}\right) = 199.48$$

Solution 2

This time we'll use the second *ds*. So, we'll first need to solve the equation for *x*. We'll also go ahead and get the derivative and root while we're at it.

$$x = y^3 \qquad\qquad \frac{dx}{dy} = 3y^2$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + 9y^4}$$

The surface area is then,

$$S = \int_1^2 2\pi x\sqrt{1 + 9y^4}\,dy$$

We used the original *y* limits this time because we picked up a *dy* from the *ds*. Also note that the presence of the *dy* means that this time, unlike the first solution, we'll need to substitute in for the *x*. Doing that gives,

$$S = \int_1^2 2\pi y^3\sqrt{1 + 9y^4}\,dy \qquad\qquad u = 1 + 9y^4$$

$$= \frac{\pi}{18}\int_{10}^{145} \sqrt{u}\,du$$

$$= \frac{\pi}{27}\left(145^{\frac{3}{2}} - 10^{\frac{3}{2}}\right) = 199.48$$

Note that after the substitution the integral was identical to the first solution and so the work was skipped.

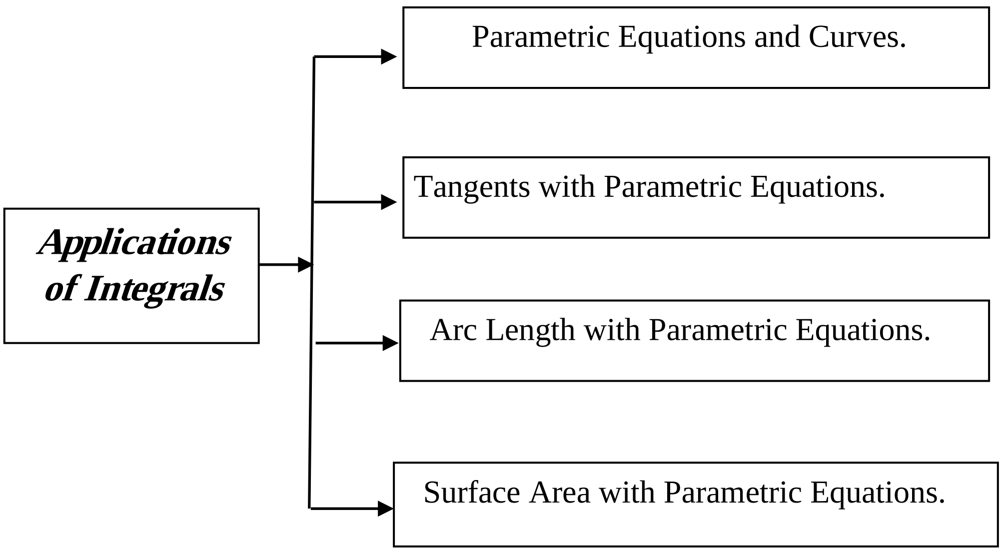
First year/ 2nd Semester - 2018-2019- Chemical and Petroleum Engineering Department

By

Ms.C. Yasir R. Al-hamdany

Lecture – Seven

Applications of Integrals



2- ***Parametric Equations and Curves*.**

To this point (in both Calculus I and Calculus II) we've looked almost exclusively at functions in the form $y = f(x)$ or $x = h(y)$ and almost all of the formulas that we've developed require that functions be in one of these two forms. The problem is that not all curves or equations that we'd like to look at fall easily into this form.

Take, for example, a circle. It is easy enough to write down the equation of a circle centered at the origin with radius *r*.

$$x^2 + y^2 = r^2$$

However, we will never be able to write the equation of a circle down as a single equation in either of the forms above. Sure we can solve for *x* or *y* as the following two formulas show

$$y = \pm\sqrt{r^2 - x^2} \qquad\qquad x = \pm\sqrt{r^2 - y^2}$$

but there are in fact two functions in each of these. Each formula gives a portion of the circle.

$$y = \sqrt{r^2 - x^2} \quad (\text{top}) \qquad\qquad x = \sqrt{r^2 - y^2} \quad (\text{right side})$$

$$y = -\sqrt{r^2 - x^2} \quad (\text{bottom}) \qquad\qquad x = -\sqrt{r^2 - y^2} \quad (\text{left side})$$

There are also a great many curves out there that we can't even write down as a single equation in terms of only *x* and *y*. So, to deal with some of these problems we introduce **parametric equations**. Instead of defining *y* in terms of *x* ($y = f(x)$) or *x* in terms of *y* ($x = h(y)$) we define both *x* and *y* in terms of a third variable called a parameter as follows,

$$x = f(t) \qquad\qquad y = g(t)$$

Each value of *t* defines a point $(x, y) = (f(t), g(t))$ that we can plot. The collection of points that we get by letting *t* be all possible values is the graph of the parametric equations and is called the **parametric curve**.

Example 1 Sketch the parametric curve for the following set of parametric equations.

$$x = t^2 + t \qquad\qquad y = 2t - 1$$

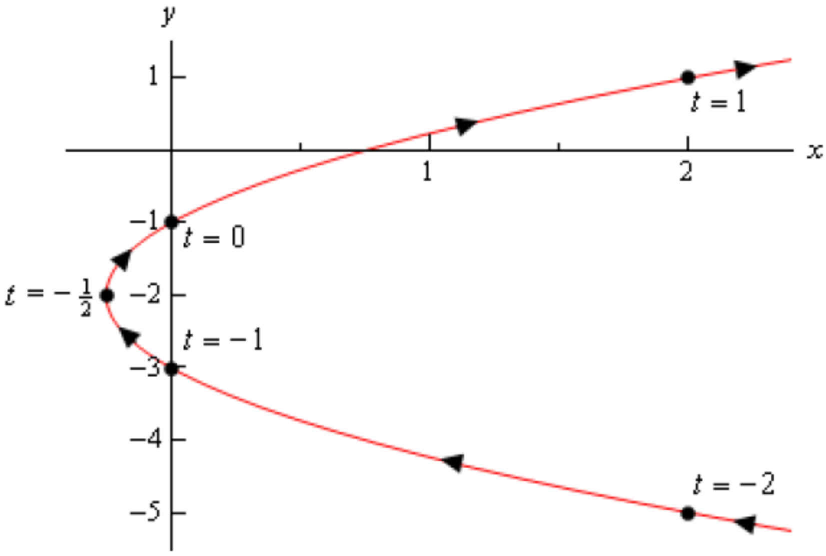
Solution

At this point our only option for sketching a parametric curve is to pick values of *t*, plug them into the parametric equations and then plot the points. So, let's plug in some *t*'s.

t	x	y
-2	2	-5
-1	0	-3
$-\frac{1}{2}$	$-\frac{1}{4}$	-2
0	0	-1
1	2	1

We have one more idea to discuss before we actually sketch the curve. Parametric curves have a **direction of motion**. The direction of motion is given by increasing *t*. So, when plotting parametric curves we also include arrows that show the direction of motion.

Here is the sketch of this parametric curve.

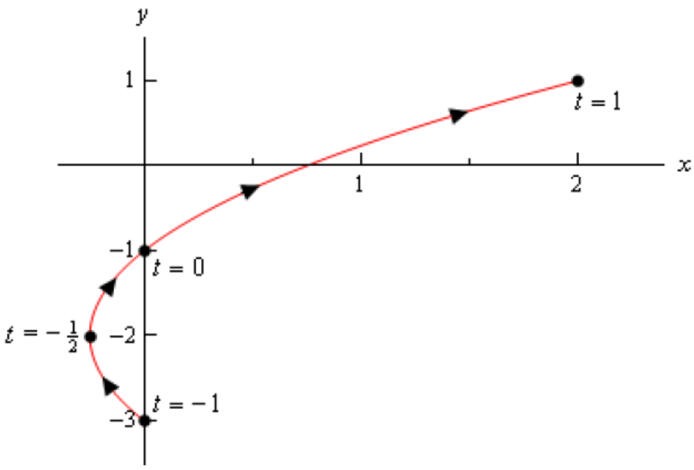


Example 2 Sketch the parametric curve for the following set of parametric equations.

$$x = t^2 + t \qquad y = 2t - 1 \qquad -1 \le t \le 1$$

Solution

Note that the only difference here is the presence of the limits on *t*. All these limits do is tell us that we can't take any value of *t* outside of this range. Therefore, the parametric curve will only be a portion of the curve above. Here is the parametric curve for this example.



Example 3 Sketch the parametric curve for the following set of parametric equations. Clearly indicate direction of motion.

$$x = 5\cos t \qquad y = 2\sin t \qquad 0 \le t \le 2\pi$$

Solution

An alternate method that we could have used here was to solve the two parametric equations for sine and cosine as follows,

$$\cos t = \frac{x}{5} \qquad \sin t = \frac{y}{2}$$

Then, recall the trig identity we used above and these new equation we get,

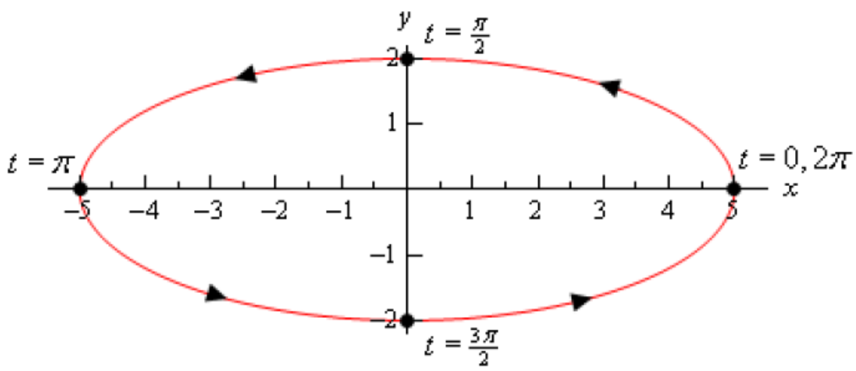
$$1 = \cos^2 t + \sin^2 t = \left(\frac{x}{5}\right)^2 + \left(\frac{y}{2}\right)^2 = \frac{x^2}{25} + \frac{y^2}{4}$$

So, here is a table of values for this set of parametric equations.

t	x	y
0	5	0
$\frac{\pi}{2}$	0	2
π	-5	0
$\frac{3\pi}{2}$	0	-2
2π	5	0

It looks like we are moving in a counter-clockwise direction about the ellipse and it also looks like we'll make exactly one complete trace of the ellipse in the range given.

Here is a sketch of the parametric curve.



Example 4

The path of a particle is given by the following set of parametric equations.

$$x = 3\cos(2t) \qquad y = 1 + \cos^2(2t)$$

Completely describe the path of this particle. Do this by sketching the path, determining limits on x and y and giving a range of t's for which the path will be traced out exactly once (provide it traces out more than once of course).

Solution

Eliminating the parameter this time will be a little different. We only have cosines this time and we'll use that to our advantage. We can solve the x equation for cosine and plug that into the equation for y. This gives,

$$\cos(2t) = \frac{x}{3} \qquad\qquad y = 1 + \left(\frac{x}{3}\right)^2 = 1 + \frac{x^2}{9}$$

This time we've got a parabola that opens upward. We also have the following limits on x and y.

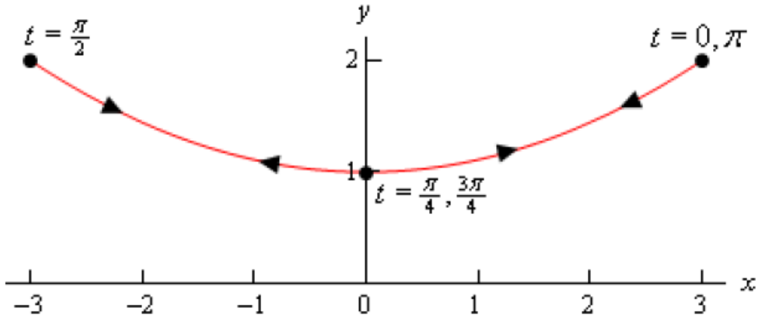
$$-1 \le \cos(2t) \le 1 \qquad -3 \le 3\cos(2t) \le 3 \qquad -3 \le x \le 3$$

$$0 \le \cos^2(2t) \le 1 \qquad 1 \le 1 + \cos^2(2t) \le 2 \qquad 1 \le y \le 2$$

So, again we only trace out a portion of the curve. Here's a set of evaluations so we can determine a range of t's for one trace of the curve.

t	x	y
0	3	2
$\frac{\pi}{4}$	0	1
$\frac{\pi}{2}$	-3	2
$\frac{3\pi}{4}$	0	1
π	3	2

So, it looks like the particle, again, will continuously trace out this portion of the curve and will make one trace in the range $0 \le t \le \frac{\pi}{2}$. Here is a sketch of the particle's path with a few value of t on it.



3- ***Tangents with Parametric Equations.***

In this section we want to find the tangent lines to the parametric equations given by,

$$x = f(t) \qquad\qquad y = g(t)$$

To do this let's first recall how to find the tangent line to $y = F(x)$ at $x = a$. Here the tangent line is given by,

$$y = F(a) + m(x-a), \quad \text{where} \quad m = \left.\frac{dy}{dx}\right|_{x=a} = F'(a)$$

Now, notice that if we could figure out how to get the derivative $\frac{dy}{dx}$ from the parametric equations we could simply reuse this formula since we will be able to use the parametric equations to find the x and y coordinates of the point.

So, just for a second let's suppose that we were able to eliminate the parameter from the parametric form and write the parametric equations in the form $y = F(x)$. Now, plug the parametric equations in for x and y. Yes, it seems silly to eliminate the parameter, then immediately put it back in, but it's what we need to do in order to get our hands on the derivative. Doing this gives,

$$g(t) = F(f(t))$$

Now, differentiate with respect to t and notice that we'll need to use the Chain Rule on the right hand side.

$$g'(t) = F'(f(t))\, f'(t)$$

Let's do another change in notation. We need to be careful with our derivatives here. Derivatives of the lower case function are with respect to t while derivatives of upper case functions are with respect to x. So, to make sure that we keep this straight let's rewrite things as follows.

$$\frac{dy}{dt} = F'(x)\frac{dx}{dt}$$

At this point we should remind ourselves just what we are after. We needed a formula for $\frac{dy}{dx}$ or $F'(x)$ that is in terms of the parametric formulas. Notice however that we can get that from the above equation.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \qquad \text{provided} \qquad \frac{dx}{dt} \neq 0$$

Derivative for Parametric Equations

$$\frac{dx}{dy} = \frac{\frac{dx}{dt}}{\frac{dy}{dt}}, \qquad \text{provided} \quad \frac{dy}{dt} \neq 0$$

Example 1 Find the tangent line(s) to the parametric curve given by

$$x = t^5 - 4t^3 \qquad\qquad y = t^2$$

at (0,4).

Solution

The first thing that we should do is find the derivative so we can get the slope of the tangent line.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{5t^4 - 12t^2} = \frac{2}{5t^3 - 12t}$$

At this point we've got a small problem. The derivative is in terms of t and all we've got is an x-y coordinate pair. The next step then is to determine the value(s) of t which will give this point. We find these by plugging the x and y values into the parametric equations and solving for t.

$$0 = t^5 - 4t^3 = t^3\left(t^2 - 4\right) \qquad \Rightarrow \qquad t = 0, \pm 2$$

$$4 = t^2 \qquad \Rightarrow \qquad t = \pm 2$$

$t = -2$

Since we already know the x and y-coordinates of the point all that we need to do is find the slope of the tangent line.

$$m = \left.\frac{dy}{dx}\right|_{t=-2} = -\frac{1}{8}$$

The tangent line (at $t = -2$) is then,

$$y = 4 - \frac{1}{8}x$$

$t = 2$

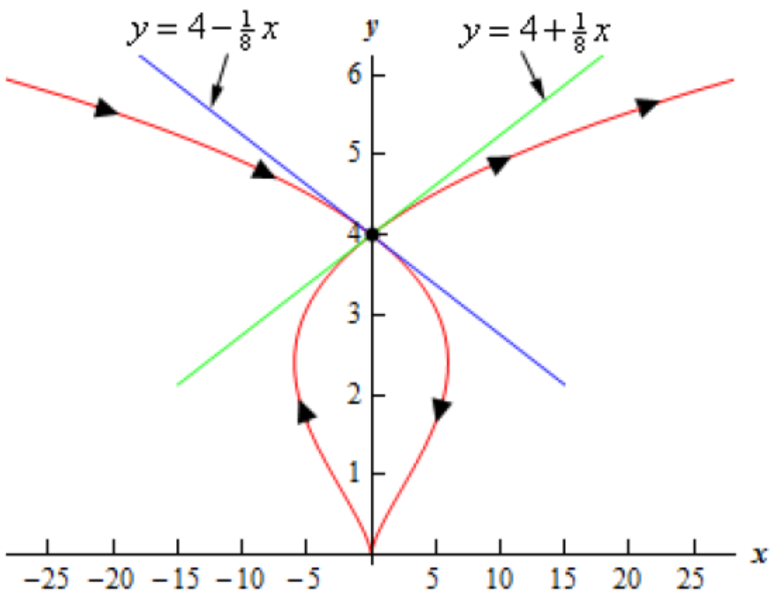
Again, all we need is the slope.

$$m = \left.\frac{dy}{dx}\right|_{t=2} = \frac{1}{8}$$

The tangent line (at $t = 2$) is then,

$$y = 4 + \frac{1}{8}x$$

A quick graph of the parametric curve will explain what is going on here.



Horizontal Tangent for Parametric Equations

$$\frac{dy}{dt}=0, \quad \text{provided} \quad \frac{dx}{dt}\neq 0$$

Vertical tangents will occur where the derivative is not defined and so we'll get vertical tangents at values of t for which we have,

Vertical Tangent for Parametric Equations

$$\frac{dx}{dt}=0, \quad \text{provided} \quad \frac{dy}{dt}\neq 0$$

Example 2 Determine the x-y coordinates of the points where the following parametric equations will have horizontal or vertical tangents.

$$x=t^3-3t \qquad\qquad y=3t^2-9$$

Solution

We'll first need the derivatives of the parametric equations.

$$\frac{dx}{dt}=3t^2-3=3\left(t^2-1\right) \qquad\qquad \frac{dy}{dt}=6t$$

Horizontal Tangents

We'll have horizontal tangents where,

$$6t=0 \qquad\qquad \Rightarrow \qquad\qquad t=0$$

Now, this is the value of t which gives the horizontal tangents and we were asked to find the x-y coordinates of the point. To get these we just need to plug t into the parametric equations. Therefore, the only horizontal tangent will occur at the point (0,-9).

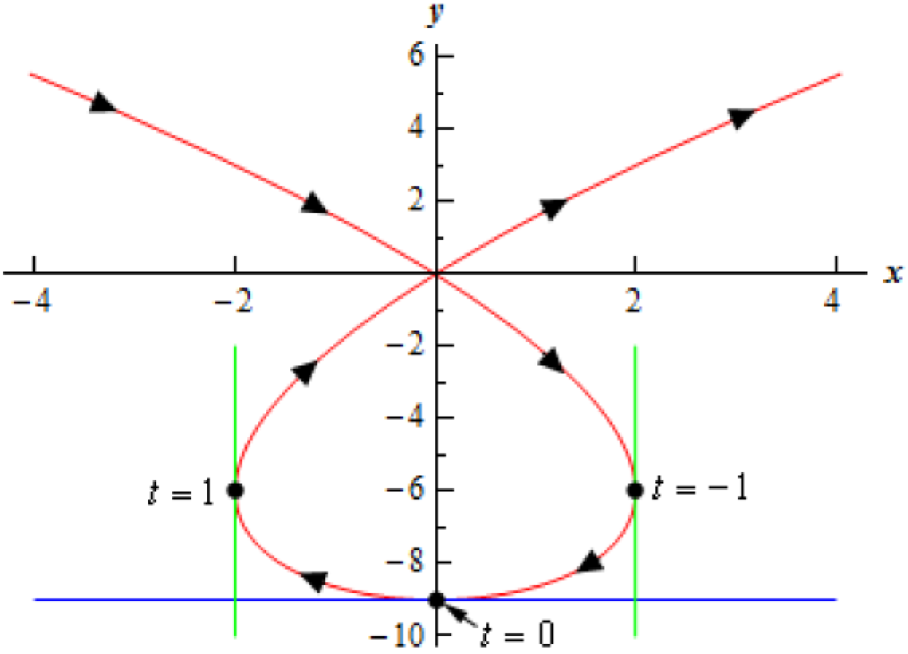
Vertical Tangents

In this case we need to solve,

$$3\left(t^2-1\right)=0 \qquad\qquad \Rightarrow \qquad\qquad t=\pm 1$$

The two vertical tangents will occur at the points (2,-6) and (-2,-6).

For the sake of completeness and at least partial verification here is the sketch of the parametric curve.



4- ***Arc Length with Parametric Equations.***

In this section we will look at the arc length of the parametric curve given by,

$$x = f(t) \qquad y = g(t) \qquad \alpha \le t \le \beta$$

We will also be assuming that the curve is traced out exactly once as t increases from α to β. We will also need to assume that the curve is traced out from left to right as t increases. This is equivalent to saying,

$$\frac{dx}{dt} \ge 0 \qquad \text{for} \quad \alpha \le t \le \beta$$

To use this we'll also need to know that,

$$dx = f'(t)\,dt = \frac{dx}{dt}dt$$

The arc length formula then becomes,

$$L = \int_\alpha^\beta \sqrt{1 + \left(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}\right)^2}\;\frac{dx}{dt}dt = \int_\alpha^\beta \sqrt{1 + \frac{\left(\frac{dy}{dt}\right)^2}{\left(\frac{dx}{dt}\right)^2}}\;\frac{dx}{dt}dt$$

$$L = \int_{\alpha}^{\beta} \frac{1}{\left|\frac{dx}{dt}\right|} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \; \frac{dx}{dt} dt$$

Now, making use of our assumption that the curve is being traced out from left to right we can drop the absolute value bars on the derivative which will allow us to cancel the two derivatives that are outside the square root and this gives,

Arc Length for Parametric Equations

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \; dt$$

Notice that we could have used the second formula for *ds* above if we had assumed instead that

$$\frac{dy}{dt} \geq 0 \qquad \text{for } \alpha \leq t \leq \beta$$

Example 1 Determine the length of the parametric curve given by the following parametric equations.

$$x = 3\sin(t) \qquad y = 3\cos(t) \qquad 0 \leq t \leq 2\pi$$

Solution

So, we can use the formula we derived above. We'll first need the following,

$$\frac{dx}{dt} = 3\cos(t) \qquad \frac{dy}{dt} = -3\sin(t)$$

The length is then,

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{9\sin^2(t) + 9\cos^2(t)} \; dt \\ &= \int_0^{2\pi} 3\sqrt{\sin^2(t) + \cos^2(t)} \; dt \\ &= 3\int_0^{2\pi} dt \\ &= 6\pi \end{aligned}$$

Example 2 Use the arc length formula for the following parametric equations.

$$x = 3\sin(3t) \qquad y = 3\cos(3t) \qquad 0 \le t \le 2\pi$$

Solution

Notice that this is the identical circle that we had in the previous example and so the length is still 6π. However, for the range given we know it will trace out the curve three times instead once as required for the formula. Despite that restriction let's use the formula anyway and see what happens.

In this case the derivatives are,

$$\frac{dx}{dt} = 9\cos(3t) \qquad \frac{dy}{dt} = -9\sin(3t)$$

and the length formula gives,

$$\begin{aligned} L &= \int_0^{2\pi} \sqrt{81\sin^2(t) + 81\cos^2(t)}\, dt \\ &= \int_0^{2\pi} 9\, dt \\ &= 18\pi \end{aligned}$$

The arc length formula can be summarized as,

$$L = \int ds$$

where,

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{if } y = f(x),\ a \le x \le b$$

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \qquad \text{if } x = h(y),\ c \le y \le d$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad \text{if } x = f(t), y = g(t),\ \alpha \le t \le \beta$$

5- ***Surface Area with Parametric Equations.***

In this final section of looking at calculus applications with parametric equations we will take a look at determining the surface area of a region obtained by rotating a parametric curve about the x or y-axis.

We will rotate the parametric curve given by,

$$x = f(t) \qquad y = g(t) \qquad \alpha \le t \le \beta$$

about the x or y-axis. We are going to assume that the curve is traced out exactly once as t increases from α to β. At this point there actually isn't all that much to do. We know that the surface area can be found by using one of the following two formulas depending on the axis of rotation (recall the Surface Area section of the Applications of Integrals chapter).

$$S = \int 2\pi y\, ds \qquad \text{rotation about } x-\text{axis}$$

$$S = \int 2\pi x\, ds \qquad \text{rotation about } y-\text{axis}$$

All that we need is a formula for ds to use and from the previous section we have,

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \qquad \text{if } x = f(t),\, y = g(t),\ \alpha \le t \le \beta$$

which is exactly what we need.

Example 1 Determine the surface area of the solid obtained by rotating the following parametric curve about the x-axis.

$$x = \cos^3\theta \qquad y = \sin^3\theta \qquad 0 \le \theta \le \frac{\pi}{2}$$

Solution

We'll first need the derivatives of the parametric equations.

$$\frac{dx}{dt} = -3\cos^2\theta\sin\theta \qquad\qquad \frac{dy}{dt} = 3\sin^2\theta\cos\theta$$

Before plugging into the surface area formula let's get the ds out of the way.

$$\begin{aligned} ds &= \sqrt{9\cos^4\theta\sin^2\theta + 9\sin^4\theta\cos^2\theta}\,dt \\ &= 3\left|\cos\theta\sin\theta\right|\sqrt{\cos^2\theta + \sin^2\theta} \\ &= 3\cos\theta\sin\theta \end{aligned}$$

Notice that we could drop the absolute value bars since both sine and cosine are positive in this range of θ given.

Now let's get the surface area and don't forget to also plug in for the y.

$$\begin{aligned} S &= \int 2\pi y\,ds \\ &= 2\pi\int_0^{\frac{\pi}{2}} \sin^3\theta\left(3\cos\theta\sin\theta\right)\,d\theta \\ &= 6\pi\int_0^{\frac{\pi}{2}} \sin^4\theta\cos\theta\,d\theta \qquad\qquad u = \sin\theta \\ &= 6\pi\int_0^1 u^4\,du \\ &= \frac{6\pi}{5} \end{aligned}$$

First year/ 2nd Semester - 2018-2019- Chemical and Petroleum Engineering Department

By

Ms.C. Yasir R. Al-hamdany

Lecture Eight

Problems &Tutorial

Problems.

A- Arc Length.

1. Set up, but do not evaluate, an integral for the length of $y = \sqrt{x+2}$, $1 \le x \le 7$ using,

(a) $ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$

(b) $ds = \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\, dy$

2. Set up, but do not evaluate, an integral for the length of $x = \cos(y)$, $0 \le x \le \frac{1}{2}$ using,

(a) $ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$

(b) $ds = \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\, dy$

3. Determine the length of $y = 7(6+x)^{\frac{3}{2}}$, $189 \le y \le 875$.

4. Determine the length of $x = 4(3+y)^2$, $1 \le y \le 4$.

B- Surface Area.

1. Set up, but do not evaluate, an integral for the surface area of the object obtained by rotating $x = \sqrt{y+5}$, $\sqrt{5} \le x \le 3$ about the y-axis using,

(a) $ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$

(b) $ds = \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\, dy$

2. Set up, but do not evaluate, an integral for the surface area of the object obtained by rotating $y = \sin(2x)$, $0 \le x \le \frac{\pi}{8}$ about the x-axis using,

(a) $ds = \sqrt{1 + \left[\frac{dy}{dx}\right]^2}\, dx$

(b) $ds = \sqrt{1 + \left[\frac{dx}{dy}\right]^2}\, dy$

3. Set up, but do not evaluate, an integral for the surface area of the object obtained by rotating $y = x^3 + 4$, $1 \le x \le 5$ about the given axis. You can use either ds.

(a) x-axis

(b) y-axis

4. Find the surface area of the object obtained by rotating $y = 4 + 3x^2$, $1 \le x \le 2$ about the y-axis.

5. Find the surface area of the object obtained by rotating $y = \sin(2x)$, $0 \le x \le \frac{\pi}{8}$ about the x-axis.

C- *Parametric Equations and Curves.*

For problems 1 – 6 eliminate the parameter for the given set of parametric equations, sketch the graph of the parametric curve and give any limits that might exist on x and y.

1. $x = 4 - 2t \qquad y = 3 + 6t - 4t^2$

2. $x = 4 - 2t \qquad y = 3 + 6t - 4t^2 \qquad 0 \le t \le 3$

3. $x = \sqrt{t+1} \qquad y = \frac{1}{t+1} \qquad t > -1$

4. $x = 3\sin(t) \qquad y = -4\cos(t) \qquad 0 \le t \le 2\pi$

5. $x = 3\sin(2t) \qquad y = -4\cos(2t) \qquad 0 \le t \le 2\pi$

6. $x = 3\sin\left(\tfrac{1}{3}t\right) \qquad y = -4\cos\left(\tfrac{1}{3}t\right) \qquad 0 \le t \le 2\pi$

For problems 7 – 11 the path of a particle is given by the set of parametric equations. Completely describe the path of the particle. To completely describe the path of the particle you will need to provide the following information.

(i) A sketch of the parametric curve (including direction of motion) based on the equation you get by eliminating the parameter.

(ii) Limits on x and y.

(iii) A range of t's for a single trace of the parametric curve.

7. $x = 3 - 2\cos(3t) \qquad y = 1 + 4\sin(3t)$

8. $x = 4\sin\left(\frac{1}{4}t\right) \qquad y = 1 - 2\cos^2\left(\frac{1}{4}t\right) \qquad -52\pi \le t \le 34\pi$

9. $x = \sqrt{4 + \cos\left(\frac{5}{2}t\right)} \qquad y = 1 + \frac{1}{3}\cos\left(\frac{5}{2}t\right) \qquad -48\pi \le t \le 2\pi$

10. $x = 2\mathbf{e}^{t} \qquad y = \cos\left(1 + \mathbf{e}^{3t}\right) \qquad 0 \le t \le \frac{3}{4}$

11. $x = \frac{1}{2}\mathbf{e}^{-3t} \qquad y = \mathbf{e}^{-6t} + 2\mathbf{e}^{-3t} - 8$

D- Tangents with Parametric Equations.

For problems 1 and 2 compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the given set of parametric equations.

1. $x = 4t^3 - t^2 + 7t \qquad y = t^4 - 6$

2. $x = \mathbf{e}^{-7t} + 2 \qquad y = 6\mathbf{e}^{2t} + \mathbf{e}^{-3t} - 4t$

For problems 3 and 4 find the equation of the tangent line(s) to the given set of parametric equations at the given point.

3. $x = 2\cos(3t) - 4\sin(3t) \qquad y = 3\tan(6t)$ at $t = \frac{\pi}{2}$

4. $x = t^2 - 2t - 11 \qquad y = t(t-4)^3 - 3t^2(t-4)^2 + 7$ at $(-3, 7)$

5. Find the values of t that will have horizontal or vertical tangent lines for the following set of parametric equations.

$$x = t^5 - 7t^4 - 3t^3 \qquad y = 2\cos(3t) + 4t$$

E- Area with Parametric Equations.

For problems 1 and 2 determine the area of the region below the parametric curve given by the set of parametric equations. For each problem you may assume that each curve traces out exactly once from right to left for the given range of t. For these problems you should only use the given parametric equations to determine the answer.

1. $x = 4t^3 - t^2 \qquad y = t^4 + 2t^2 \qquad 1 \le t \le 3$

2. $x = 3 - \cos^3(t) \qquad y = 4 + \sin(t) \qquad 0 \le t \le \pi$

F- Arc Length with Parametric Equations.

For problems 1 and 2 determine the length of the parametric curve given by the set of parametric equations. For these problems you may assume that the curve traces out exactly once for the given range of t's.

1. $x = 8t^{\frac{3}{2}}$ $\quad y = 3 + (8-t)^{\frac{3}{2}}$ $\quad 0 \le t \le 4$

2. $x = 3t+1$ $\quad y = 4 - t^2$ $\quad -2 \le t \le 0$

3. A particle travels along a path defined by the following set of parametric equations. Determine the total distance the particle travels and compare this to the length of the parametric curve itself.

$$x = 4\sin\left(\tfrac{1}{4}t\right) \qquad y = 1 - 2\cos^2\left(\tfrac{1}{4}t\right) \qquad -52\pi \le t \le 34\pi$$

For problems 4 and 5 set up, but do not evaluate, an integral that gives the length of the parametric curve given by the set of parametric equations. For these problems you may assume that the curve traces out exactly once for the given range of t's.

4. $x = 2 + t^2$ $\quad y = \mathbf{e}^t \sin(2t)$ $\quad 0 \le t \le 3$

5. $x = \cos^3(2t)$ $\quad y = \sin(1 - t^2)$ $\quad -\frac{3}{2} \le t \le 0$

G- <u>Surface Area with Parametric Equations.</u>

For problems1 – 3 determine the surface area of the object obtained by rotating the parametric curve about the given axis. For these problems you may assume that the curve traces out exactly once for the given range of t's.

1. Rotate $x = 3 + 2t$ $\quad y = 9 - 3t$ $\quad 1 \le t \le 4$ about the y-axis.

2. Rotate $x = 9 + 2t^2$ $\quad y = 4t$ $\quad 0 \le t \le 2$ about the x-axis.

3. Rotate $x = 3\cos(\pi t)$ $\quad y = 5t + 2$ $\quad 0 \le t \le \frac{1}{2}$ about the y-axis.

For problems 4 and 5 set up, but do not evaluate, an integral that gives the surface area of the object obtained by rotating the parametric curve about the given axis. For these problems you may assume that the curve traces out exactly once for the given range of t's.

4. Rotate $x = 1 + \ln(5 + t^2)$ $\quad y = 2t - 2t^2$ $\quad 0 \le t \le 2$ about the x-axis.

5. Rotate $x = 1 + 3t^2$ $\quad y = \sin(2t)\cos\left(\tfrac{1}{4}t\right)$ $\quad 0 \le t \le \frac{1}{2}$ about the y-axis.