First year/ 2nd Semester - 2018-2019- Chemical and Petroleum Engineering Department

By

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Lecture – One

Integration Techniques:-

1- *Integration by Parts.*

2- *Integrals Involving Trig Functions.*

Integration Techniques:-

1- ***Integration by Parts***

So let's derive the integration by parts formula. We'll start with the product rule.

$$(f\,g)' = f'\,g + f\,g'$$

Now, integrate both sides of this.

$$\int (f\,g)'\,dx = \int f'\,g + f\,g'\,dx$$

The left side is easy enough to integrate and we'll split up the right side of the integral.

$$fg = \int f'\,g\,dx + \int f\,g'\,dx$$

Finally, rewrite the formula as follows and we arrive at the integration by parts formula.

$$\int f\,g'\,dx = fg - \int f'\,g\,dx$$

This is not the easiest formula to use however. So, let's do a couple of substitutions.

$$u = f(x) \qquad v = g(x)$$

$$du = f'(x)\,dx \qquad dv = g'(x)\,dx$$

Using these substitutions gives us the formula that most people think of as the integration by parts formula.

$$\int u\,dv = uv - \int v\,du$$

To use this formula we will need to identify u and dv, compute du and v and then use the formula. Note as well that computing v is very easy. All we need to do is integrate dv.

$$v = \int dv$$

Example 1 Evaluate the following integral

$$\int x\mathbf{e}^{6x}\,dx$$

Solution

$$u = x \qquad dv = \mathbf{e}^{6x}\,dx$$

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$$du = dx \qquad v = \int \mathbf{e}^{6x}\,dx = \frac{1}{6}\mathbf{e}^{6x}$$

Next, let's take a look at integration by parts for definite integrals. The integration by parts formula for definite integrals is,

Integration by Parts, Definite Integrals

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Example 2 Evaluate the following integral.

$$\int_{-1}^{2} x\mathbf{e}^{6x}\,dx$$

Solution
This is the same integral that we looked at in the first example so we'll use the same u and dv to get,

$$\begin{aligned}\int_{-1}^{2} x\mathbf{e}^{6x}\,dx &= \frac{x}{6}\mathbf{e}^{6x}\Big|_{-1}^{2} - \frac{1}{6}\int_{-1}^{2}\mathbf{e}^{6x}\,dx \\ &= \frac{x}{6}\mathbf{e}^{6x}\Big|_{-1}^{2} - \frac{1}{36}\mathbf{e}^{6x}\Big|_{-1}^{2} \\ &= \frac{11}{36}\mathbf{e}^{12} + \frac{7}{36}\mathbf{e}^{-6}\end{aligned}$$

Example 3 Evaluate the following integral.

$$\int (3t+5)\cos\left(\frac{t}{4}\right)dt$$

Solution
Instead of splitting the integral up let's instead use the following choices for u and dv.

$$u = 3t+5 \qquad dv = \cos\left(\frac{t}{4}\right)dt$$

$$du = 3\,dt \qquad v = 4\sin\left(\frac{t}{4}\right)$$

The integral is then,

$$\begin{aligned}\int (3t+5)\cos\left(\frac{t}{4}\right)dt &= 4(3t+5)\sin\left(\frac{t}{4}\right) - 12\int \sin\left(\frac{t}{4}\right)dt \\ &= 4(3t+5)\sin\left(\frac{t}{4}\right) + 48\cos\left(\frac{t}{4}\right) + c\end{aligned}$$

Example 4 Evaluate the following integral.

$$\int w^2 \sin(10w)\,dw$$

Solution
For this example we'll use the following choices for u and dv.

$$u = w^2 \qquad dv = \sin(10w)\,dw$$

$$du = 2w\,dw \qquad v = -\frac{1}{10}\cos(10w)$$

The integral is then,

$$\int w^2 \sin(10w)\,dw = -\frac{w^2}{10}\cos(10w) + \frac{1}{5}\int w\cos(10w)\,dw$$

In this example, unlike the previous examples, the new integral will also require integration by parts. For this second integral we will use the following choices.

$$u = w \qquad dv = \cos(10w)\,dw$$

$$du = dw \qquad v = \frac{1}{10}\sin(10w)$$

So, the integral becomes,

$$\begin{aligned}\int w^2 \sin(10w)\,dw &= -\frac{w^2}{10}\cos(10w) + \frac{1}{5}\left(\frac{w}{10}\sin(10w) - \frac{1}{10}\int \sin(10w)\,dw\right)\\ &= -\frac{w^2}{10}\cos(10w) + \frac{1}{5}\left(\frac{w}{10}\sin(10w) + \frac{1}{100}\cos(10w)\right) + c\\ &= -\frac{w^2}{10}\cos(10w) + \frac{w}{50}\sin(10w) + \frac{1}{500}\cos(10w) + c\end{aligned}$$

Example 5 Evaluate the following integral

$$\int x\sqrt{x+1}\,dx$$

(a) Using Integration by Parts. [Solution]

(b) Using a standard Calculus I substitution. [Solution]

Solution

(a) Evaluate using Integration by Parts.

In this case we'll use the following choices for u and dv.

$$u = x \qquad dv = \sqrt{x+1}\,dx$$

$$du = dx \qquad v = \frac{2}{3}(x+1)^{\frac{3}{2}}$$

The integral is then,

$$\begin{aligned}\int x\sqrt{x+1}\,dx &= \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{2}{3}\int (x+1)^{\frac{3}{2}}\,dx\\ &= \frac{2}{3}x(x+1)^{\frac{3}{2}} - \frac{4}{15}(x+1)^{\frac{5}{2}} + c\end{aligned}$$

(b) Evaluate Using a standard Calculus I substitution.

Now let's do the integral with a substitution. We can use the following substitution.

$$u = x + 1 \qquad x = u - 1 \qquad du = dx$$

Notice that we'll actually use the substitution twice, once for the quantity under the square root and once for the x in front of the square root. The integral is then,

$$\begin{aligned}\int x\sqrt{x+1}\,dx &= \int (u-1)\sqrt{u}\,du \\ &= \int u^{\frac{3}{2}} - u^{\frac{1}{2}}\,du \\ &= \frac{2}{5}u^{\frac{5}{2}} - \frac{2}{3}u^{\frac{3}{2}} + c \\ &= \frac{2}{5}(x+1)^{\frac{5}{2}} - \frac{2}{3}(x+1)^{\frac{3}{2}} + c\end{aligned}$$

Example 6 Evaluate the following integral.

$$\int \mathbf{e}^{\theta}\cos\theta\,d\theta$$

Solution

$$u = \cos\theta \qquad dv = \mathbf{e}^{\theta}\,d\theta$$
$$du = -\sin\theta\,d\theta \qquad v = \mathbf{e}^{\theta}$$

The integral is then,

$$\int \mathbf{e}^{\theta}\cos\theta\,d\theta = \mathbf{e}^{\theta}\cos\theta + \int \mathbf{e}^{\theta}\sin\theta\,d\theta$$

So, it looks like we'll do integration by parts again. Here are our choices this time.

$$u = \sin\theta \qquad dv = \mathbf{e}^{\theta}\,d\theta$$
$$du = \cos\theta\,d\theta \qquad v = \mathbf{e}^{\theta}$$

The integral is now,

$$\int \mathbf{e}^{\theta}\cos\theta\,d\theta = \mathbf{e}^{\theta}\cos\theta + \mathbf{e}^{\theta}\sin\theta - \int \mathbf{e}^{\theta}\cos\theta\,d\theta$$

$$2\int \mathbf{e}^{\theta}\cos\theta\,d\theta = \mathbf{e}^{\theta}\cos\theta + \mathbf{e}^{\theta}\sin\theta$$

All we need to do now is divide by 2 and we're done. The integral is,

$$\int \mathbf{e}^{\theta}\cos\theta\,d\theta = \frac{1}{2}\left(\mathbf{e}^{\theta}\cos\theta + \mathbf{e}^{\theta}\sin\theta\right) + c$$

2- ***Integrals Involving Trig Functions***.

Let's start off with an integral that we should already be able to do.

$$\int \cos x \sin^5 x\,dx = \int u^5\,du \qquad \text{using the substitution } u = \sin x$$
$$= \frac{1}{6}\sin^6 x + c$$

Example 1 Evaluate the following integral.

$$\int \sin^5 x\,dx$$

Solution

This integral no longer has the cosine in it that would allow us to use the substitution that we used above. Therefore, that substitution won't work and we are going to have to find another way of doing this integral.

Let's first notice that we could write the integral as follows,

$$\int \sin^5 x\,dx = \int \sin^4 x \sin x\,dx = \int \left(\sin^2 x\right)^2 \sin x\,dx$$

Now recall the trig identity,

$$\cos^2 x + \sin^2 x = 1 \qquad \Rightarrow \qquad \sin^2 x = 1 - \cos^2 x$$

With this identity the integral can be written as,

$$\int \sin^5 x\,dx = \int \left(1-\cos^2 x\right)^2 \sin x\,dx$$

and we can now use the substitution $u = \cos x$. Doing this gives us,

$$\int \sin^5 x\,dx = -\int \left(1-u^2\right)^2 du$$
$$= -\int 1 - 2u^2 + u^4\,du$$
$$= -\left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right) + c$$
$$= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c$$

The exponent on the remaining sines will then be even and we can easily convert the remaining sines to cosines using the identity,

$$\cos^2 x + \sin^2 x = 1 \qquad (1)$$

Example 2 Evaluate the following integral.

$$\int \sin^6 x \cos^3 x\,dx$$

Solution

So, in this case we've got both sines and cosines in the problem and in this case the exponent on the sine is even while the exponent on the cosine is odd. So, we can use a similar technique in this integral. This time we'll strip out a cosine and convert the rest to sines.

$$\int \sin^6 x \cos^3 x\,dx = \int \sin^6 x \cos^2 x \cos x\,dx$$

$$= \int \sin^6 x \left(1 - \sin^2 x\right) \cos x \, dx \qquad\qquad u = \sin x$$

$$= \int u^6 \left(1 - u^2\right) du$$

$$= \int u^6 - u^8 \, du$$

$$= \frac{1}{7}\sin^7 x - \frac{1}{9}\sin^9 x + c$$

At this point let's pause for a second to summarize what we've learned so far about integrating powers of sine and cosine.

$$\int \sin^n x \cos^m x \, dx \tag{2}$$

The integrals involving products of sines and cosines in which both exponents are even can be done using one or more of the following formulas to rewrite the integrand.

$$\cos^2 x = \frac{1}{2}\left(1 + \cos\left(2x\right)\right)$$

$$\sin^2 x = \frac{1}{2}\left(1 - \cos\left(2x\right)\right)$$

$$\sin x \cos x = \frac{1}{2}\sin\left(2x\right)$$

Example 3 Evaluate the following integral.

$$\int \sin^2 x \cos^2 x \, dx$$

Solution

Solution 1

In this solution we will use the two half angle formulas above and just substitute them into the integral.

$$\int \sin^2 x \cos^2 x \, dx = \int \frac{1}{2}\left(1 - \cos\left(2x\right)\right)\left(\frac{1}{2}\right)\left(1 + \cos\left(2x\right)\right) dx$$

$$= \frac{1}{4}\int 1 - \cos^2\left(2x\right) dx$$

In fact to eliminate the remaining problem term all that we need to do is reuse the first half angle formula given above.

$$\begin{aligned}\int \sin^2 x\cos^2 x\,dx &= \frac{1}{4}\int 1-\frac{1}{2}\left(1+\cos\left(4x\right)\right)dx\\ &= \frac{1}{4}\int \frac{1}{2}-\frac{1}{2}\cos\left(4x\right)dx\\ &= \frac{1}{4}\left(\frac{1}{2}x-\frac{1}{8}\sin\left(4x\right)\right)+c\\ &= \frac{1}{8}x-\frac{1}{32}\sin\left(4x\right)+c\end{aligned}$$

Solution 2

In this solution we will use the half angle formula to help simplify the integral as follows.

$$\begin{aligned}\int \sin^2 x\cos^2 x\,dx &= \int\left(\sin x\cos x\right)^2 dx\\ &= \int\left(\frac{1}{2}\sin\left(2x\right)\right)^2 dx\\ &= \frac{1}{4}\int \sin^2\left(2x\right)dx\end{aligned}$$

Now, we use the double angle formula for sine to reduce to an integral that we can do.

$$\begin{aligned}\int \sin^2 x\cos^2 x\,dx &= \frac{1}{8}\int 1-\cos\left(4x\right)dx\\ &= \frac{1}{8}x-\frac{1}{32}\sin\left(4x\right)+c\end{aligned}$$

Sometimes in the process of reducing integrals in which both exponents are even we will run across products of sine and cosine in which the arguments are different. These will require one of the following formulas to reduce the products to integrals that we can do.

$$\sin\alpha\cos\beta = \frac{1}{2}\left[\sin\left(\alpha-\beta\right)+\sin\left(\alpha+\beta\right)\right]$$

$$\sin\alpha\sin\beta = \frac{1}{2}\left[\cos\left(\alpha-\beta\right)-\cos\left(\alpha+\beta\right)\right]$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos\left(\alpha-\beta\right)+\cos\left(\alpha+\beta\right)\right]$$

Example 4 Evaluate the following integral.

$$\int \cos\left(15x\right)\cos\left(4x\right)dx$$

Solution

This integral requires the last formula listed above.

$$\begin{aligned}\int \cos\left(15x\right)\cos\left(4x\right)dx &= \frac{1}{2}\int \cos\left(11x\right)+\cos\left(19x\right)dx\\ &= \frac{1}{2}\left(\frac{1}{11}\sin\left(11x\right)+\frac{1}{19}\sin\left(19x\right)\right)+c\end{aligned}$$

It's now time to look at integrals that involve products of secants and tangents. This time, let's do a little analysis of the possibilities before we just jump into examples. The general integral will be,

$$\int \sec^n x \tan^m x \, dx \qquad (3)$$

The first thing to notice is that we can easily convert even powers of secants to tangents and even powers of tangents to secants by using a formula similar to (1). In fact, the formula can be derived from (1) so let's do that.

$$\sin^2 x + \cos^2 x = 1$$

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$\tan^2 x + 1 = \sec^2 x \qquad (4)$$

Now, we're going to want to deal with (3) similarly to how we dealt with (2). We'll want to eventually use one of the following substitutions.

$$u = \tan x \qquad du = \sec^2 x \, dx$$

$$u = \sec x \qquad du = \sec x \tan x \, dx$$

Example 5 Evaluate the following integral.

$$\int \sec^9 x \tan^5 x \, dx$$

Solution

First note that since the exponent on the secant isn't even we can't use the substitution $u = \tan x$. However, the exponent on the tangent is odd and we've got a secant in the integral and so we will be able to use the substitution $u = \sec x$. This means stripping out a single tangent (along with a secant) and converting the remaining tangents to secants using (4).

Here's the work for this integral.

$$\begin{aligned}\int \sec^9 x \tan^5 x \, dx &= \int \sec^8 x \tan^4 x \tan x \sec x \, dx \\ &= \int \sec^8 x \left(\sec^2 x - 1\right)^2 \tan x \sec x \, dx \qquad u = \sec x \\ &= \int u^8 \left(u^2 - 1\right)^2 du \\ &= \int u^{12} - 2u^{10} + u^8 \, du \\ &= \frac{1}{13}\sec^{13} x - \frac{2}{11}\sec^{11} x + \frac{1}{9}\sec^9 x + c\end{aligned}$$

Example 6 Evaluate the following integral.

$$\int \sec^4 x \tan^6 x \, dx$$

Solution

So, in this example the exponent on the tangent is even so the substitution $u = \sec x$ won't work. The exponent on the secant is even and so we can use the substitution $u = \tan x$ for this integral. That means that we need to strip out two secants and convert the rest to tangents. Here is the work for this integral.

$$\begin{aligned}
\int \sec^4 x \tan^6 x \, dx &= \int \sec^2 x \tan^6 x \sec^2 x \, dx \\
&= \int \left(\tan^2 x + 1\right) \tan^6 x \sec^2 x \, dx \qquad u = \tan x \\
&= \int \left(u^2 + 1\right) u^6 \, du \\
&= \int u^8 + u^6 \, du \\
&= \frac{1}{9}\tan^9 x + \frac{1}{7}\tan^7 x + c
\end{aligned}$$

Example 7 Evaluate the following integral.

$$\int \tan x \, dx$$

Solution

To do this integral all we need to do is recall the definition of tangent in terms of sine and cosine and then this integral is nothing more than a Calculus I substitution.

$$\begin{aligned}
\int \tan x \, dx &= \int \frac{\sin x}{\cos x} dx \qquad u = \cos x \\
&= -\int \frac{1}{u} du \\
&= -\ln\left|\cos x\right| + c \qquad r \ln x = \ln x^r \\
&= \ln\left|\cos x\right|^{-1} + c \\
&\ln\left|\sec x\right| + c
\end{aligned}$$

Example 8 Evaluate the following integral.

$$\int \tan^3 x \, dx$$

Solution

The trick to this one is do the following manipulation of the integrand.

$$\begin{aligned}
\int \tan^3 x \, dx &= \int \tan x \tan^2 x \, dx \\
&= \int \tan x \left(\sec^2 x - 1\right) dx \\
&= \int \tan x \sec^2 x \, dx - \int \tan x \, dx
\end{aligned}$$

We can now use the substitution $u = \tan x$ on the first integral and the results from the previous example on the second integral.

The integral is then,

$$\int \tan^3 x \, dx = \frac{1}{2}\tan^2 x - \ln\left|\sec x\right| + c$$

Note that all odd powers of tangent (with the exception of the first power) can be integrated using the same method we used in the previous example. For instance,

$$\int \tan^5 x \, dx = \int \tan^3 x \left(\sec^2 x - 1\right) dx = \int \tan^3 x \sec^2 x \, dx - \int \tan^3 x \, dx$$

Example 9 Evaluate the following integral.

$$\int \sec x \, dx$$

Solution

This one isn't too bad once you see what you've got to do. By itself the integral can't be done. However, if we manipulate the integrand as follows we can do it.

$$\begin{aligned} \int \sec x \, dx &= \int \frac{\sec x \left(\sec x + \tan x \right)}{\sec x + \tan x} dx \\ &= \int \frac{\sec^2 x + \tan x \sec x}{\sec x + \tan x} dx \end{aligned}$$

In this form we can do the integral using the substitution $u = \sec x + \tan x$. Doing this gives,

$$\int \sec x \, dx = \ln \left| \sec x + \tan x \right| + c$$

Example 10 Evaluate the following integral.

$$\int \sec^3 x \, dx$$

Solution

This one is different from any of the other integrals that we've done in this section. The first step to doing this integral is to perform integration by parts using the following choices for u and dv.

$$u = \sec x \qquad\qquad dv = \sec^2 x \, dx$$
$$du = \sec x \tan x \, dx \qquad\qquad v = \tan x$$

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec x \tan^2 x \, dx$$

To do this integral we'll first write the tangents in the integral in terms of secants. Again, this is not necessarily an obvious choice but it's what we need to do in this case.

$$\begin{aligned} \int \sec^3 x \, dx &= \sec x \tan x - \int \sec x \left(\sec^2 x - 1 \right) dx \\ &= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx \end{aligned}$$

the first integral is exactly the integral we're being asked to evaluate with a minus sign in front. So, add it to both sides to get,

$$2 \int \sec^3 x \, dx = \sec x \tan x + \ln \left| \sec x + \tan x \right|$$

Finally divide by two and we're done.

$$\int \sec^3 x \, dx = \frac{1}{2} \left(\sec x \tan x + \ln \left| \sec x + \tan x \right| \right) + c$$

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Lecture – Two

Integration Techniques:-

3- Integrals Involving Partial Fractions.

4- Integrals Involving Roots.

5- ***Integrals Involving Partial Fractions***.

let's start this section out with an integral that we can already do so we can contrast it with the integrals that we'll be doing in this section.

$$\int \frac{2x-1}{x^2-x-6}dx = \int \frac{1}{u}du \qquad \text{using} \quad u = x^2-x-6 \quad \text{and} \quad du = (2x-1)dx$$

$$= \ln\left|x^2-x-6\right| + c$$

So, if the numerator is the derivative of the denominator (or a constant multiple of the derivative of the denominator) doing this kind of integral is fairly simple. However, often the numerator isn't the derivative of the denominator (or a constant multiple). For example, consider the following integral.

$$\int \frac{3x+11}{x^2-x-6}dx$$

This process of taking a rational expression and decomposing it into simpler rational expressions that we can add or subtract to get the original rational expression is called **partial fraction decomposition**. Many integrals involving rational expressions can be done if we first do partial fractions on the integrand.

So, let's do a quick review of partial fractions. We'll start with a rational expression in the form,

$$f(x) = \frac{P(x)}{Q(x)}$$

where both *P(x)* and *Q(x)* are polynomials and the degree of *P(x)* is smaller than the degree of *Q(x)*. Recall that the degree of a polynomial is the largest exponent in the polynomial. Partial fractions can only be done if the degree of the numerator is strictly less than the degree of the denominator. That is important to remember.

So, once we've determined that partial fractions can be done we factor the denominator as completely as possible. Then for each factor in the denominator we can use the following table to determine the term(s) we pick up in the partial fraction decomposition.

Factor in denominator	Term in partial fraction decomposition
$ax+b$	$\dfrac{A}{ax+b}$
$(ax+b)^k$	$\dfrac{A_1}{ax+b}+\dfrac{A_2}{(ax+b)^2}+\cdots+\dfrac{A_k}{(ax+b)^k}$, $k=1,2,3,\ldots$
ax^2+bx+c	$\dfrac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\dfrac{A_1x+B_1}{ax^2+bx+c}+\dfrac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\dfrac{A_kx+B_k}{(ax^2+bx+c)^k}$, $k=1,2,3,\ldots$

There are several methods for determining the coefficients for each term and we will go over each of those in the following examples.

Let's start the examples by doing the integral above.

Example 1 Evaluate the following integral.

$$\int \frac{3x+11}{x^2-x-6}\,dx$$

Solution

The first step is to factor the denominator as much as possible and get the form of the partial fraction decomposition. Doing this gives,

$$\frac{3x+11}{(x-3)(x+2)}=\frac{A}{x-3}+\frac{B}{x+2}$$

The next step is to actually add the right side back up.

$$\frac{3x+11}{(x-3)(x+2)}=\frac{A(x+2)+B(x-3)}{(x-3)(x+2)}$$

Now, we need to choose A and B so that the numerators of these two are equal for every x. To do this we'll need to set the numerators equal.

$$3x+11=A(x+2)+B(x-3)$$

What we're going to do here is to notice that the numerators must be equal for *any* x that we would choose to use. In particular the numerators must be equal for $x=-2$ and $x=3$. So, let's plug these in and see what we get.

$$x=-2 \qquad 5=A(0)+B(-5) \qquad \Rightarrow \qquad B=-1$$

$$x=3 \qquad 20=A(5)+B(0) \qquad \Rightarrow \qquad A=4$$

At this point there really isn't a whole lot to do other than the integral.

$$\begin{aligned}\int \frac{3x+11}{x^2-x-6}\,dx &= \int \frac{4}{x-3}-\frac{1}{x+2}\,dx \\ &= \int \frac{4}{x-3}\,dx-\int \frac{1}{x+2}\,dx \\ &= 4\ln|x-3|-\ln|x+2|+c\end{aligned}$$

There is also another integral that often shows up in these kinds of problems so we may as well give the formula for it here since we are already on the subject.

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$$

Example 2 Evaluate the following integral.

$$\int \frac{x^2+4}{3x^3+4x^2-4x}\,dx$$

Solution

We won't be putting as much detail into this solution as we did in the previous example. The first thing is to factor the denominator and get the form of the partial fraction decomposition.

$$\frac{x^2+4}{x(x+2)(3x-2)} = \frac{A}{x}+\frac{B}{x+2}+\frac{C}{3x-2}$$

The next step is to set numerators equal. If you need to actually add the right side together to get

$$x^2+4 = A(x+2)(3x-2)+Bx(3x-2)+Cx(x+2)$$

As with the previous example it looks like we can just pick a few values of x and find the constants so let's do that.

$$x=0 \qquad 4=A(2)(-2) \qquad \Rightarrow \qquad A=-1$$

$$x=-2 \qquad 8=B(-2)(-8) \qquad \Rightarrow \qquad B=\frac{1}{2}$$

$$x=\frac{2}{3} \qquad \frac{40}{9}=C\left(\frac{2}{3}\right)\left(\frac{8}{3}\right) \qquad \Rightarrow \qquad C=\frac{40}{16}=\frac{5}{2}$$

Now, let's do the integral.

$$\int \frac{x^2+4}{3x^3+4x^2-4x}\,dx = \int -\frac{1}{x}+\frac{\frac{1}{2}}{x+2}+\frac{\frac{5}{2}}{3x-2}\,dx$$

$$= -\ln|x|+\frac{1}{2}\ln|x+2|+\frac{5}{6}\ln|3x-2|+c$$

Example 3 Evaluate the following integral.

$$\int \frac{x^2-29x+5}{(x-4)^2(x^2+3)}\,dx$$

Solution

This time the denominator is already factored so let's just jump right to the partial fraction decomposition.

$$\frac{x^2-29x+5}{(x-4)^2(x^2+3)} = \frac{A}{x-4}+\frac{B}{(x-4)^2}+\frac{Cx+D}{x^2+3}$$

Setting numerators gives,

$$x^2-29x+5 = A(x-4)(x^2+3)+B(x^2+3)+(Cx+D)(x-4)^2$$

In this case we aren't going to be able to just pick values of x that will give us all the constants. Therefore, we will need to work this the second (and often longer) way. The first step is to multiply out the right side and collect all the like terms together. Doing this gives,

$$x^2-29x+5=(A+C)x^3+(-4A+B-8C+D)x^2+(3A+16C-8D)x-12A+3B+16D$$

In other words we will need to set the coefficients of like powers of x equal. This will give a system of equations that can be solved.

$$\left.\begin{aligned} x^3: &\quad A+C=0\\ x^2: &\quad -4A+B-8C+D=1\\ x^1: &\quad 3A+16C-8D=-29\\ x^0: &\quad -12A+3B+16D=5 \end{aligned}\right\} \quad\Rightarrow\quad A=1, B=-5, C=-1, D=2$$

Now, let's take a look at the integral.

$$\begin{aligned}\int \frac{x^2-29x+5}{(x-4)^2(x^2+3)}\,dx &= \int \frac{1}{x-4}-\frac{5}{(x-4)^2}+\frac{-x+2}{x^2+3}\,dx\\ &= \int \frac{1}{x-4}-\frac{5}{(x-4)^2}-\frac{x}{x^2+3}+\frac{2}{x^2+3}\,dx\\ &= \ln|x-4|+\frac{5}{x-4}-\frac{1}{2}\ln|x^2+3|+\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)+c\end{aligned}$$

Example 4 Evaluate the following integral.

$$\int \frac{x^3+10x^2+3x+36}{(x-1)(x^2+4)^2}\,dx$$

Solution

Let's first get the general form of the partial fraction decomposition.

$$\frac{x^3+10x^2+3x+36}{(x-1)(x^2+4)^2}=\frac{A}{x-1}+\frac{Bx+C}{x^2+4}+\frac{Dx+E}{(x^2+4)^2}$$

Now, set numerators equal, expand the right side and collect like terms.

$$\begin{aligned}x^3+10x^2+3x+36 &= A(x^2+4)^2+(Bx+C)(x-1)(x^2+4)+(Dx+E)(x-1)\\ &= (A+B)x^4+(C-B)x^3+(8A+4B-C+D)x^2+\\ &\qquad (-4B+4C-D+E)x+16A-4C-E\end{aligned}$$

Setting coefficient equal gives the following system.

$$\left.\begin{aligned} x^4: &\quad A+B=0\\ x^3: &\quad C-B=1\\ x^2: &\quad 8A+4B-C+D=10\\ x^1: &\quad -4B+4C-D+E=3\\ x^0: &\quad 16A-4C-E=36 \end{aligned}\right\} \quad\Rightarrow\quad A=2, B=-2, C=-1, D=1, E=0$$

Here's the integral.

$$\int \frac{x^3+10x^2+3x+36}{(x-1)(x^2+4)^2}\,dx = \int \frac{2}{x-1}+\frac{-2x-1}{x^2+4}+\frac{x}{(x^2+4)^2}\,dx$$

$$= \int \frac{2}{x-1}-\frac{2x}{x^2+4}-\frac{1}{x^2+4}+\frac{x}{(x^2+4)^2}\,dx$$

$$= 2\ln|x-1| - \ln|x^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}\frac{1}{x^2+4}+c$$

To this point we've only looked at rational expressions where the degree of the numerator was strictly less that the degree of the denominator. Of course not all rational expressions will fit into this form and so we need to take a look at a couple of examples where this isn't the case.

If a rational function $\dfrac{R(x)}{Q(x)}$ is such that the degree of $R(x)$ is **greater than** the degree of $Q(x)$, then one must use long division and write the rational function in the form

$$\frac{R(x)}{Q(x)} = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n + \frac{P(x)}{Q(x)}$$

where now $P(x)$ is a remainder term with the degree of $P(x)$ **less than** the degree of $Q(x)$ and our object is to integrate each term of the above representation.

Example 5 Evaluate the following integral.

$$\int \frac{x^4-5x^3+6x^2-18}{x^3-3x^2}\,dx$$

Solution

So, in this case the degree of the numerator is 4 and the degree of the denominator is 3. Therefore, partial fractions can't be done on this rational expression.

To fix this up we'll need to do long division on this to get it into a form that we can deal with. Here is the work for that.

$$\begin{array}{r|l} & x-2 \\ \hline x^3-3x^2 & x^4-5x^3+6x^2-18 \\ & -(x^4-3x^3) \\ \hline & -2x^3+6x^2-18 \\ & -(-2x^3+6x^2) \\ \hline & -18 \end{array}$$

$$\frac{x^4 - 5x^3 + 6x^2 - 18}{x^3 - 3x^2} = x - 2 - \frac{18}{x^3 - 3x^2}$$

and the integral becomes,

$$\int \frac{x^4 - 5x^3 + 6x^2 - 18}{x^3 - 3x^2}\, dx = \int x - 2 - \frac{18}{x^3 - 3x^2}\, dx$$

$$= \int x - 2\, dx - \int \frac{18}{x^3 - 3x^2}\, dx$$

The first integral we can do easily enough and the second integral is now in a form that allows us to do partial fractions. So, let's get the general form of the partial fractions for the second integrand.

$$\frac{18}{x^2(x-3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-3}$$

Setting numerators equal gives us,

$$18 = Ax(x-3) + B(x-3) + Cx^2$$

$x = 0$	$18 = B(-3)$	$\Rightarrow$	$B = -6$
$x = 3$	$18 = C(9)$	$\Rightarrow$	$C = 2$
$x = 1$	$18 = A(-2) + B(-2) + C = -2A + 14$	$\Rightarrow$	$A = -2$

The integral is then,

$$\int \frac{x^4 - 5x^3 + 6x^2 - 18}{x^3 - 3x^2}\, dx = \int x - 2\, dx - \int -\frac{2}{x} - \frac{6}{x^2} + \frac{2}{x-3}\, dx$$

$$= \frac{1}{2}x^2 - 2x + 2\ln|x| - \frac{6}{x} - 2\ln|x-3| + c$$

6- ***Integrals Involving Roots.***

Example 1 Evaluate the following integral.

$$\int \frac{x+2}{\sqrt[3]{x-3}}\,dx$$

Solution

Sometimes when faced with an integral that contains a root we can use the following substitution to simplify the integral into a form that can be easily worked with.

$$u = \sqrt[3]{x-3}$$

So, instead of letting u be the stuff under the radical as we often did in Calculus I we let u be the whole radical. Now, there will be a little more work here since we will also need to know what x is so we can substitute in for that in the numerator and so we can compute the differential, dx. This is easy enough to get however. Just solve the substitution for x as follows,

$$x = u^3 + 3 \qquad\qquad dx = 3u^2\,du$$

Using this substitution the integral is now,

$$\begin{aligned}\int \frac{\left(u^3+3\right)+2}{u}3u^2du &= \int 3u^4 + 15u\,du \\ &= \frac{3}{5}u^5 + \frac{15}{2}u^2 + c \\ &= \frac{3}{5}(x-3)^{\frac{5}{3}} + \frac{15}{2}(x-3)^{\frac{2}{3}} + c\end{aligned}$$

So, sometimes, when an integral contains the root $\sqrt[n]{g(x)}$ the substitution,

$$u = \sqrt[n]{g(x)}$$

can be used to simplify the integral into a form that we can deal with.

Example 2 Evaluate the following integral.

$$\int \frac{2}{x - 3\sqrt{x+10}}\,dx$$

Solution

We'll do the same thing we did in the previous example. Here's the substitution and the extra work we'll need to do to get x in terms of u.

$$u = \sqrt{x+10} \qquad x = u^2 - 10 \qquad dx = 2u\,du$$

With this substitution the integral is,

$$\int \frac{2}{x - 3\sqrt{x+10}}\,dx = \int \frac{2}{u^2 - 10 - 3u}(2u)\,du = \int \frac{4u}{u^2 - 3u - 10}\,du$$

This integral can now be done with partial fractions.

$$\frac{4u}{(u-5)(u+2)} = \frac{A}{u-5} + \frac{B}{u+2}$$

Setting numerators equal gives,

$$4u = A(u+2) + B(u-5)$$

Picking value of u gives the coefficients.

$$u = -2 \qquad -8 = B(-7) \qquad B = \frac{8}{7}$$

$$u = 5 \qquad 20 = A(7) \qquad A = \frac{20}{7}$$

The integral is then,

$$\begin{aligned} \int \frac{2}{x - 3\sqrt{x+10}}\,dx &= \int \frac{\frac{20}{7}}{u-5} + \frac{\frac{8}{7}}{u+2}\,du \\ &= \frac{20}{7}\ln|u-5| + \frac{8}{7}\ln|u+2| + c \\ &= \frac{20}{7}\ln\left|\sqrt{x+10} - 5\right| + \frac{8}{7}\ln\left|\sqrt{x+10} + 2\right| + c \end{aligned}$$

First year/ 2nd Semester - 2018-2019- Chemical and Petroleum Engineering Department

By

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Lecture – Three

Integration Techniques

Tutorials & Problems

A- ## ***Integration by Parts***

Evaluate each of the following integrals.

1. $\int 4x\cos(2-3x)\,dx$

2. $\int_6^0 (2+5x)\mathbf{e}^{\frac{1}{3}x}\,dx$

3. $\int (3t+t^2)\sin(2t)\,dt$

4. $\int 6\tan^{-1}\left(\frac{8}{w}\right)dw$

5. $\int \mathbf{e}^{2z}\cos\left(\frac{1}{4}z\right)dz$

6. $\int_0^{\pi} x^2\cos(4x)\,dx$

7. $\int t^7\sin(2t^4)\,dt$

8. $\int y^6\cos(3y)\,dy$

9. $\int (4x^3-9x^2+7x+3)\mathbf{e}^{-x}\,dx$

B-

Integrals Involving Trig Functions

Evaluate each of the following integrals.

1. $\int \sin^3\left(\frac{2}{3}x\right)\cos^4\left(\frac{2}{3}x\right)dx$

2. $\int \sin^8\left(3z\right)\cos^5\left(3z\right)dz$

3. $\int \cos^4\left(2t\right)dt$

4. $\int_{\pi}^{2\pi} \cos^3\left(\frac{1}{2}w\right)\sin^5\left(\frac{1}{2}w\right)dw$

5. $\int \sec^6\left(3y\right)\tan^2\left(3y\right)dy$

6. $\int \tan^3\left(6x\right)\sec^{10}\left(6x\right)dx$

7. $\int_{0}^{\frac{\pi}{4}} \tan^7\left(z\right)\sec^3\left(z\right)dz$

8. $\int \cos\left(3t\right)\sin\left(8t\right)dt$

C- ***Partial Fractions***

Evaluate each of the following integrals.

1. $\int \frac{4}{x^2+5x-14}\,dx$

2. $\int \frac{8-3t}{10t^2+13t-3}\,dt$

3. $\int_{-1}^{0} \frac{w^2+7w}{(w+2)(w-1)(w-4)}\,dw$

4. $\int \frac{8}{3x^3+7x^2+4x}\,dx$

5. $\int_{2}^{4} \frac{3z^2+1}{(z+1)(z-5)^2}\,dz$

6. $\int \frac{4x-11}{x^3-9x^2}\,dx$

7. $\int \frac{z^2+2z+3}{(z-6)(z^2+4)}\,dz$

8. $\int \frac{8+t+6t^2-12t^3}{(3t^2+4)(t^2+7)}\,dt$

D-

Integrals Involving Roots

Evaluate each of the following integrals.

1. $\displaystyle\int \frac{7}{2+\sqrt{x-4}}\,dx$

2. $\displaystyle\int \frac{1}{w+2\sqrt{1-w}+2}\,dw$

3. $\displaystyle\int \frac{t-2}{t-3\sqrt{2t-4}+2}\,dt$