First year/ 2nd Semester - 2018-2019- Chemical and Petroleum Engineering Department

By

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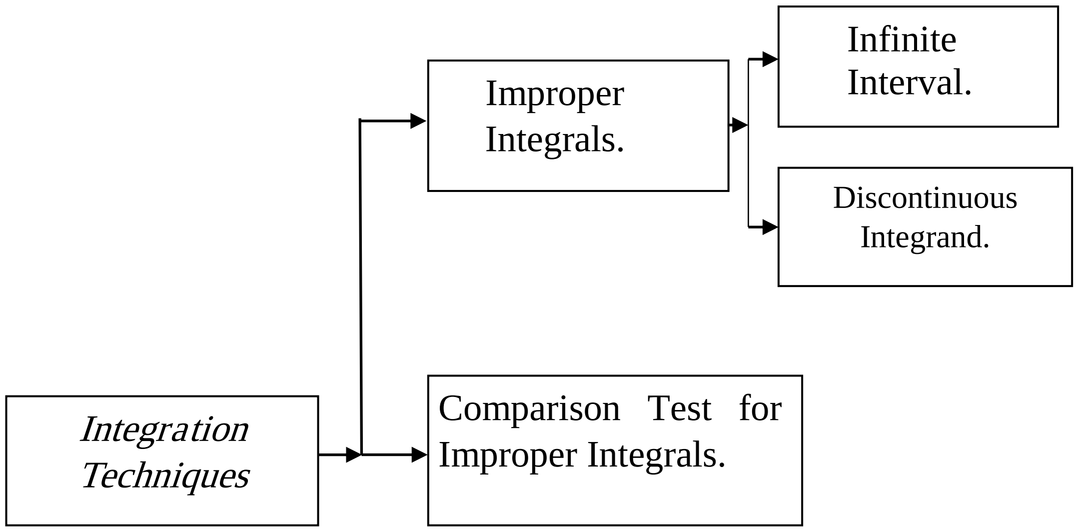
Lecture – Four

Integration Techniques



1- ***Improper Integrals*.**

A- Infinite Interval.

In this kind of integral one or both of the limits of integration are infinity. In these cases the interval of integration is said to be over an infinite interval.

Example 1 Evaluate the following integral.

$$\int_1^{\infty} \frac{1}{x^2}\,dx$$

Solution.

To see how we're going to do this integral let's think of this as an area problem. So instead of asking what the integral is, let's instead ask what the area under $f(x) = \frac{1}{x^2}$ on the interval $[1,\infty)$ is.

We still aren't able to do this, however, let's step back a little and instead ask what the area under $f(x)$ is on the interval $[1,t]$ where $t > 1$ and t is finite. This is a problem that we can do.

$$A_t = \int_1^t \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg|_1^t = 1 - \frac{1}{t}$$

Now, we can get the area under $f(x)$ on $[1,\infty)$ simply by taking the limit of A_t as t goes to infinity.

$$A = \lim_{t\to\infty} A_t = \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1$$

This is then how we will do the integral itself.

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^2}\,dx &= \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx \\ &= \lim_{t\to\infty}\left(-\frac{1}{x}\right)\bigg|_1^t \\ &= \lim_{t\to\infty}\left(1 - \frac{1}{t}\right) = 1\end{aligned}$$

Let's now get some definitions out of the way. We will call these integrals **convergent** if the associated limit exists and is a finite number (*i.e.* it's not plus or minus infinity) and **divergent** if the associated limit either doesn't exist or is (plus or minus) infinity.

Let's now formalize up the method for dealing with infinite intervals. There are essentially three cases that we'll need to look at.

1. If $\int_a^t f(x)\,dx$ exists for every $t > a$ then,

$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty}\int_a^t f(x)\,dx$$

provided the limit exists and is finite.

2. If $\int_t^b f(x)\,dx$ exists for every $t < b$ then,

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t \to -\infty} \int_t^b f(x)\,dx$$

provided the limits exists and is finite.

3. If $\int_{-\infty}^{c} f(x)\,dx$ and $\int_c^{\infty} f(x)\,dx$ are both convergent then,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx$$

Example 2 Determine if the following integral is convergent or divergent and if it's convergent find its value.

$$\int_1^{\infty} \frac{1}{x}\,dx$$

Solution

So, the first thing we do is convert the integral to a limit.

$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{t \to \infty} \int_1^t \frac{1}{x}\,dx$$

Now, do the integral and the limit.

$$\begin{aligned}\int_1^{\infty} \frac{1}{x}\,dx &= \lim_{t \to \infty} \ln(x)\Big|_1^t \\ &= \lim_{t \to \infty} \left(\ln(t) - \ln 1\right) \\ &= \infty\end{aligned}$$

So, the limit is infinite and so the integral is divergent.

Fact

If $a > 0$ then

$$\int_a^{\infty} \frac{1}{x^p}\,dx$$

is convergent if $p > 1$ and divergent if $p \le 1$.

Example 3 Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-\infty}^{0} \frac{1}{\sqrt{3-x}}\,dx$$

Solution

There really isn't much to do with these problems once you know how to do them. We'll convert the integral to a limit/integral pair, evaluate the integral and then the limit.

$$\begin{aligned}
\int_{-\infty}^{0} \frac{1}{\sqrt{3-x}}\,dx &= \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{\sqrt{3-x}}\,dx \\
&= \lim_{t\to-\infty} -2\sqrt{3-x}\Big|_{t}^{0} \\
&= \lim_{t\to-\infty} \left(-2\sqrt{3} + 2\sqrt{3-t}\right) \\
&= -2\sqrt{3} + \infty \\
&= \infty
\end{aligned}$$

So, the limit is infinite and so this integral is divergent.

Example 4 Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-\infty}^{\infty} x\mathbf{e}^{-x^2}\,dx$$

Solution

In this case we've got infinities in both limits and so we'll need to split the integral up into two separate integrals. We can split the integral up at any point, so let's choose $a = 0$ since this will be a convenient point for the evaluation process. The integral is then,

$$\int_{-\infty}^{\infty} x\mathbf{e}^{-x^2}\,dx = \int_{-\infty}^{0} x\mathbf{e}^{-x^2}\,dx + \int_{0}^{\infty} x\mathbf{e}^{-x^2}\,dx$$

We've now got to look at each of the individual limits.

$$\begin{aligned}
\int_{-\infty}^{0} x\mathbf{e}^{-x^2}\,dx &= \lim_{t\to-\infty} \int_{t}^{0} x\mathbf{e}^{-x^2}\,dx \\
&= \lim_{t\to-\infty} \left(-\frac{1}{2}\mathbf{e}^{-x^2}\right)\Bigg|_{t}^{0} \\
&= \lim_{t\to-\infty} \left(-\frac{1}{2} + \frac{1}{2}\mathbf{e}^{-t^2}\right) \\
&= -\frac{1}{2}
\end{aligned}$$

So, the first integral is convergent. Note that this does NOT mean that the second integral will also be convergent. So, let's take a look at that one.

$$\int_{0}^{\infty} x\mathbf{e}^{-x^2}\,dx = \lim_{t\to\infty} \int_{0}^{t} x\mathbf{e}^{-x^2}\,dx$$

$$= \lim_{t\to\infty} \left(-\frac{1}{2}\mathbf{e}^{-x^2} \right)\Big|_0^t$$

$$= \lim_{t\to\infty} \left(-\frac{1}{2}\mathbf{e}^{-t^2} + \frac{1}{2} \right)$$

$$= \frac{1}{2}$$

This integral is convergent and so since they are both convergent the integral we were actually asked to deal with is also convergent and its value is,

$$\int_{-\infty}^{\infty} x\mathbf{e}^{-x^2}\,dx = \int_{-\infty}^{0} x\mathbf{e}^{-x^2}\,dx + \int_{0}^{\infty} x\mathbf{e}^{-x^2}\,dx = -\frac{1}{2} + \frac{1}{2} = 0$$

Example 5 Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-2}^{\infty} \sin x\,dx$$

Solution

First convert to a limit.

$$\int_{-2}^{\infty} \sin x\,dx = \lim_{t\to\infty} \int_{-2}^{t} \sin x\,dx$$

$$= \lim_{t\to\infty} \left(-\cos x\right)\Big|_{-2}^{t}$$

$$= \lim_{t\to\infty} \left(\cos 2 - \cos t\right)$$

This limit doesn't exist and so the integral is divergent.

B- <u>Discontinuous Integrand.</u>

We now need to look at the second type of improper integrals that we'll be looking at in this section. These are integrals that have discontinuous integrands. The process here is basically the same with one subtle difference. Here are the general cases that we'll look at for these integrals.

1. If $f(x)$ is continuous on the interval $[a,b)$ and not continuous at $x=b$ then,

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

provided the limit exists and is finite. Note as well that we do need to use a left hand limit here since the interval of integration is entirely on the left side of the upper limit.

2. If $f(x)$ is continuous on the interval $(a,b]$ and not continuous at $x=a$ then,

$$\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

provided the limit exists and is finite. In this case we need to use a right hand limit here since the interval of integration is entirely on the right side of the lower limit.

3. If $f(x)$ is not continuous at $x = c$ where $a < c < b$ and $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are both convergent then,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

As with the infinite interval case this requires BOTH of the integrals to be convergent in order for this integral to also be convergent. If either of the two integrals is divergent then so is this integral.

4. If $f(x)$ is not continuous at $x = a$ and $x = b$ and if $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are both convergent then,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Where c is any number. Again, this requires BOTH of the integrals to be convergent in order for this integral to also be convergent.

Example 6 Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_0^3 \frac{1}{\sqrt{3-x}}\,dx$$

Solution

The problem point is the upper limit so we are in the first case above.

$$\begin{aligned}\int_0^3 \frac{1}{\sqrt{3-x}}\,dx &= \lim_{t\to 3^-}\int_0^t \frac{1}{\sqrt{3-x}}\,dx \\ &= \lim_{t\to 3^-}\left(-2\sqrt{3-x}\right)\Big|_0^t \\ &= \lim_{t\to 3^-}\left(2\sqrt{3} - 2\sqrt{3-t}\right) \\ &= 2\sqrt{3}\end{aligned}$$

The limit exists and is finite and so the integral converges and the integral's value is $2\sqrt{3}$.

Example 7 Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{-2}^3 \frac{1}{x^3}\,dx$$

Solution

This integrand is not continuous at $x = 0$ and so we'll need to split the integral up at that point.

$$\int_{-2}^3 \frac{1}{x^3}\,dx = \int_{-2}^0 \frac{1}{x^3}\,dx + \int_0^3 \frac{1}{x^3}\,dx$$

Now we need to look at each of these integrals and see if they are convergent.

$$\int_{-2}^{0} \frac{1}{x^3}\,dx = \lim_{t\to 0^-}\int_{-2}^{t} \frac{1}{x^3}\,dx$$
$$= \lim_{t\to 0^-}\left(-\frac{1}{2x^2}\right)\Bigg|_{-2}^{t}$$
$$= \lim_{t\to 0^-}\left(-\frac{1}{2t^2}+\frac{1}{8}\right)$$
$$= -\infty$$

At this point we're done. One of the integrals is divergent that means the integral that we were asked to look at is divergent. We don't even need to bother with the second integral.

Example 8 Determine if the following integral is convergent or divergent. If it is convergent find its value.

$$\int_{0}^{\infty} \frac{1}{x^2}\,dx$$

Solution

This is an integral over an infinite interval that also contains a discontinuous integrand. To do this integral we'll need to split it up into two integrals. We can split it up anywhere, but pick a value that will be convenient for evaluation purposes.

$$\int_{0}^{\infty} \frac{1}{x^2}\,dx = \int_{0}^{1} \frac{1}{x^2}\,dx + \int_{1}^{\infty} \frac{1}{x^2}\,dx$$

In order for the integral in the example to be convergent we will need BOTH of these to be convergent. If one or both are divergent then the whole integral will also be divergent.

We know that the second integral is convergent by the fact given in the infinite interval portion above. So, all we need to do is check the first integral.

$$\int_{0}^{1} \frac{1}{x^2}\,dx = \lim_{t\to 0^+}\int_{t}^{1} \frac{1}{x^2}\,dx$$
$$= \lim_{t\to 0^+}\left(-\frac{1}{x}\right)\Bigg|_{t}^{1}$$
$$= \lim_{t\to 0^+}\left(-1+\frac{1}{t}\right)$$
$$= \infty$$

2- ***Comparison Test for Improper Integrals*.**

Now that we've seen how to actually compute improper integrals we need to address one more topic about them. Often we aren't concerned with the actual value of these integrals. Instead we might only be interested in whether the integral is convergent or divergent. Also, there will be some integrals that we simply won't be able to integrate and yet we would still like to know if they converge or diverge.

To deal with this we've got a test for convergence or divergence that we can use to help us answer the question of convergence for an improper integral.

Comparison Test

If $f(x) \geq g(x) \geq 0$ on the interval $[a, \infty)$ then,

1. If $\int_a^\infty f(x)\,dx$ converges then so does $\int_a^\infty g(x)\,dx$.
2. If $\int_a^\infty g(x)\,dx$ diverges then so does $\int_a^\infty f(x)\,dx$.

Note that if you think in terms of area the Comparison Test makes a lot of sense. If $f(x)$ is larger than $g(x)$ then the area under $f(x)$ must also be larger than the area under $g(x)$.

So, if the area under the larger function is finite (*i.e.* $\int_a^\infty f(x)\,dx$ converges) then the area under the smaller function must also be finite (*i.e.* $\int_a^\infty g(x)\,dx$ converges). Likewise, if the area under the smaller function is infinite (*i.e.* $\int_a^\infty g(x)\,dx$ diverges) then the area under the larger function must also be infinite (*i.e.* $\int_a^\infty f(x)\,dx$ diverges).

Example 1 Determine if the following integral is convergent or divergent.

$$\int_2^\infty \frac{\cos^2 x}{x^2}\,dx$$

Solution.

So, it seems like it would be nice to have some idea as to whether the integral converges or diverges ahead of time so we will know whether we will need to look for a larger (and convergent) function or a smaller (and divergent) function.

Therefore, it seems likely that the denominator will determine the convergence/divergence of this integral and we know that

$$\int_2^\infty \frac{1}{x^2}\,dx$$

converges since $p = 2 > 1$ by the fact in the previous section. So let's guess that this integral will converge.

So we now know that we need to find a function that is larger than

$$\frac{\cos^2 x}{x^2}$$

and also converges. Making a fraction larger is actually a fairly simple process. We can either make the numerator larger or we can make the denominator smaller. In this case we can't do a lot about the denominator. However we can use the fact that $0 \le \cos^2 x \le 1$ to make the numerator larger (*i.e.* we'll replace the cosine with something we know to be larger, namely 1). So,

$$\frac{\cos^2 x}{x^2} \le \frac{1}{x^2}$$

Now, as we've already noted

$$\int_2^\infty \frac{1}{x^2}\,dx$$

converges and so by the Comparison Test we know that

$$\int_2^\infty \frac{\cos^2 x}{x^2}\,dx$$

must also converge.

Example 2 Determine if the following integral is convergent or divergent.

$$\int_3^\infty \frac{1}{x + \mathbf{e}^x}\,dx$$

Solution

Let's first take a guess about the convergence of this integral. As noted after the fact in the last section about

The question then is which one to drop? Let's first drop the exponential. Doing this gives,

$$\frac{1}{x + \mathbf{e}^x} < \frac{1}{x}$$

This is a problem however, since

$$\int_3^\infty \frac{1}{x}\,dx$$

diverges by the fact. We've got a larger function that is divergent. This doesn't say anything about the smaller function. Therefore, we chose the wrong one to drop.

Let's try it again and this time let's drop the x.

$$\frac{1}{x + \mathbf{e}^x} < \frac{1}{\mathbf{e}^x} = \mathbf{e}^{-x}$$

Also,

$$\begin{aligned}\int_3^\infty \mathbf{e}^{-x}\,dx &= \lim_{t\to\infty}\int_3^t \mathbf{e}^{-x}\,dx \\ &= \lim_{t\to\infty}\left(-\mathbf{e}^{-t}+\mathbf{e}^{-3}\right) \\ &= \mathbf{e}^{-3}\end{aligned}$$

So, $\int_3^\infty \mathbf{e}^{-x}\,dx$ is convergent. Therefore, by the Comparison test

$$\int_3^\infty \frac{1}{x+\mathbf{e}^{x}}\,dx$$

is also convergent.

Example 3 Determine if the following integral is convergent or divergent.

$$\int_3^\infty \frac{1}{x-\mathbf{e}^{-x}}\,dx$$

Solution

This is where the second change will come into play. As before we know that both x and the exponential are positive. However, this time since we are subtracting the exponential from the x if we were to drop the exponential the denominator will become larger and so the fraction will become smaller. In other words,

$$\frac{1}{x-\mathbf{e}^{-x}} > \frac{1}{x}$$

and we know that

$$\int_3^\infty \frac{1}{x}\,dx$$

diverges and so by the Comparison Test we know that

$$\int_3^\infty \frac{1}{x-\mathbf{e}^{-x}}\,dx$$

must also diverge.

Example 4 Determine if the following integral is convergent or divergent.

$$\int_1^\infty \frac{1+3\sin^4(2x)}{\sqrt{x}}\,dx$$

Solution

Therefore, since the exponent on the denominator is less than 1 we can guess that the integral will probably diverge. We will need a smaller function that also diverges.

We know that $0 \le \sin^4(2x) \le 1$. In particular, this term is positive and so if we drop it from the numerator the numerator will get smaller. This gives,

$$\frac{1+3\sin^4(2x)}{\sqrt{x}} > \frac{1}{\sqrt{x}}$$

and

$$\int_1^\infty \frac{1}{\sqrt{x}}\,dx$$

diverges so by the Comparison Test

$$\int_1^\infty \frac{1+3\sin^4(2x)}{\sqrt{x}}\,dx$$

also diverges.

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Lecture – Five

Integration Techniques

Tutorials & Problems-2

Problems.

Improper Integrals

Determine if each of the following integrals converge or diverge. If the integral converges determine its value.

1. $\int_{0}^{\infty} (1+2x)\mathbf{e}^{-x}\,dx$

2. $\int_{-\infty}^{0} (1+2x)\mathbf{e}^{-x}\,dx$

3. $\int_{-5}^{1} \frac{1}{10+2z}\,dz$

4. $\int_{1}^{2} \frac{4w}{\sqrt[3]{w^2-4}}\,dw$

5. $\int_{-\infty}^{1} \sqrt{6-y}\,dy$

6. $\int_{2}^{\infty} \frac{9}{(1-3z)^4}\,dz$

7. $\int_{0}^{4} \frac{x}{x^2-9}\,dx$

8. $\int_{-\infty}^{\infty} \frac{6w^3}{\left(w^4+1\right)^2}\,dw$

9. $\int_{1}^{4} \frac{1}{x^2+x-6}\,dx$

10. $\int_{-\infty}^{0} \frac{\mathbf{e}^{\frac{1}{x}}}{x^2}\,dx$

Comparison Test for Improper Integrals

Use the Comparison Test to determine if the following integrals converge or diverge.

1. $\int_1^{\infty} \frac{1}{x^3+1}\,dx$

2. $\int_3^{\infty} \frac{z^2}{z^3-1}\,dz$

3. $\int_4^{\infty} \frac{\mathbf{e}^{-y}}{y}\,dy$

4. $\int_1^{\infty} \frac{z-1}{z^4+2z^2}\,dz$