1.1 Function and their Graphs

Functions are a toll for describing the real world in mathematical terms. A function can be represented by an equation, a graph, a numerical table, or a verbal description. The area of a circle depends on the radius of the circle; the distance an object travels at constant speed along a straight-line path depends on the elapsed time. In each case, the value of one variable quantity, say **y** , depends on the value of another variable quantity, which we might call **x** . we say that " **y** is a function of **x** " and write this symbolically as

$$y = f(x)$$

f represents the function

x independent variable representing the input value of f

y dependent variable or output value of f at x

Definition

A function f from a set D to a set Y is a rule that assigns a unique (single) element $f(x) \in Y$ to each element $x \in D$.

The set D of all possible input values is called the **domain** of the function. The set of all values of $f(x)$ as **x** varies throughout **D** is called the **range** of the function. The range may not include every element in the set Y. the domain and range of a function can be any sets of objects or sets of real numbers interpreted as points of a coordinate line.

Such as equations

$A = \pi r^2$ $\qquad$ $R = mv / r^2$ $\qquad$ $y = vt - 1/2at^2$

Where A, R, y are functions $f(x)$ and r, t are variables.

<u>Example 1 :</u>

Let's verify the natural domains and associated ranges of some simple functions. The domains in each case are the values of **x** for which the formula makes sense.

Function	Domain(x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1`]$	$[0, 1`]$

<u>Solution :</u>

The formula $y = x^2$ gives areal y-values for any real no. of x

D(x) : $(-\infty, \infty)$

R(y) : $[0, \infty)$ because the square of any real no. is non-negative and every non-negative no. y is the square of its own square root, $y = \left(\sqrt{y}\right)^2$ for $y \geq 0$

The formula $y = 1/x$ gives a real y-value for every x except x=0.

D(x) : $(-\infty, 0) \cup (0, \infty)$

R(y) : $(-\infty, 0) \cup (0, \infty)$

The formula $y = \sqrt{x}$ gives a real y-value only if x $\geq$ 0 .

D(x) : $[0, \infty)$

R(y) : $[0, \infty)$

The formula $y = \sqrt{1 - x^2}$ gives a real y-value for every x in the closed interval from (-1) to (+1), outside this domain, $(1 - x^2)$ is negative and its square root is not a real no. only if x $\geq$ 0 .

D(x) : $[-1, 1`]$

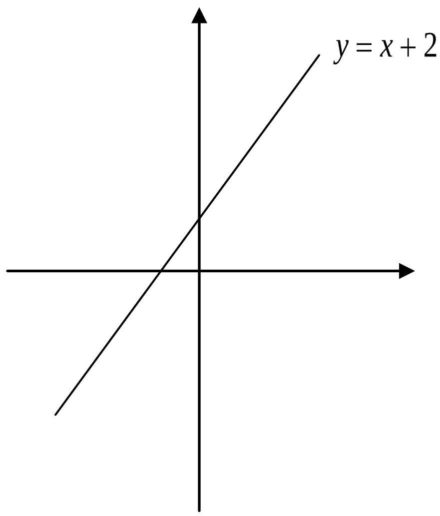
R(y) : $[0, 1`]$

1.2 Graphs of Functions :

If f is a function with domain D, its graph consists of the points in the Cartesian plane. Whose coordinates are the input-output pairs for f. in set notation, the graph is

$$\{(x, f(x)) \mid x \in D\}$$

For example; the graph of the function $f(x) = x + 2$ is the set of points with coordinates (x, y) for with $y = x + 2$



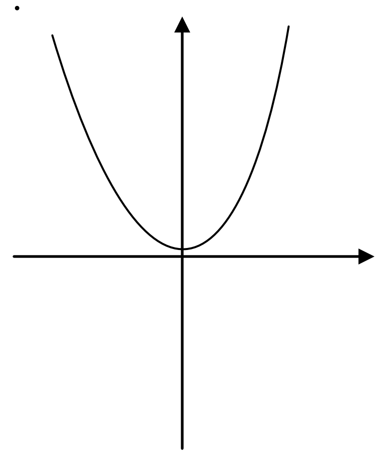
The graph of a function f is a useful picture of its behavior.

If (x, y) is a point on the graph, then $y = f(x)$

Example 2 :

Graph the function $y = x^2$ over the interval $[-2,2]$.

x	$f(x) = x^2$
-2	4
-1	1
0	0
1	1
-2	4

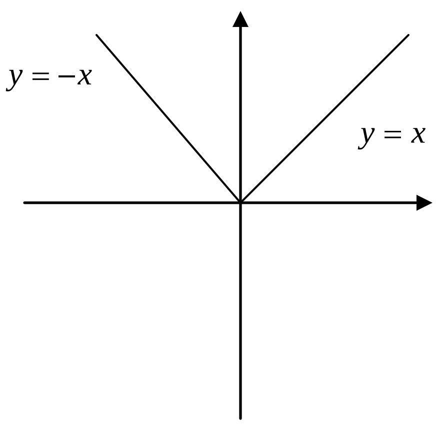


1.3 piecewise - Defined Functions :

Sometimes a function is described by using different formulas on different parts of its domain. For example; the absolute value function

$$|x| = \begin{matrix} x & x \geq 0 \\ -x & x\langle 0 \end{matrix}$$

Whose graph is given in figure below, the right-hand side of the equation means that the function equals x if $x \geq 0$, and equals $-x$ if $x\langle 0$

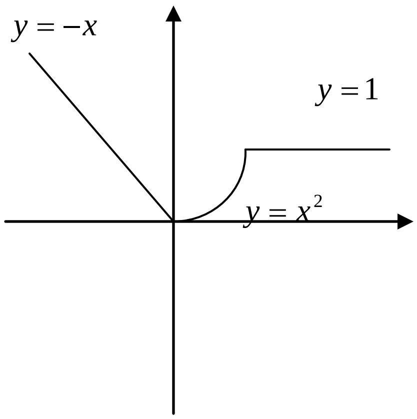


$D(x):(-\infty,\infty)$
$R(x):[0,\infty)$

Example 3 :

The function $f(x) = \begin{matrix} -x & x\langle 0 \\ x^2 & 0 \leq x \leq 1 \\ 1 & x\rangle 1 \end{matrix}$

To graph the function $y = f(x)$ shown here, we apply different formulas to different parts of its domain



<u>1.4 Even Functions and Odd functions :</u>

The graphs of even and odd functions have characteristic symmetry properties.

<u>Definition</u>

A function $y = f(x)$ is an

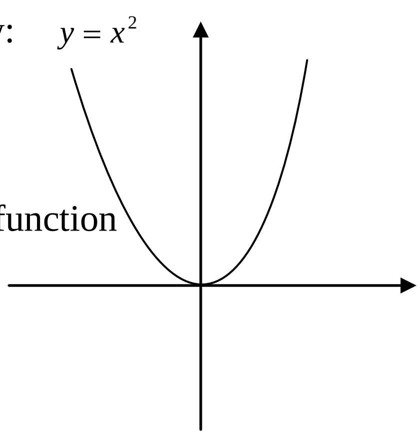
Even function of x if $f(-x) = f(x)$

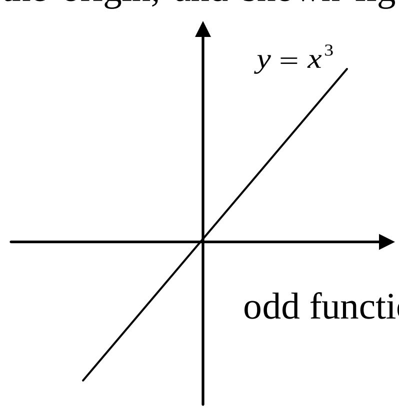
Odd function of x if $f(-x) = -f(x)$

For every x in the function's domain.

For example: $y = x^2$, $y = x^4$, it is an even function of x because $(-x)^2 = x^2$ and $(-x)^4 = x^4$, and $y = x$, $y = x^3$, it is an odd function of x because $(-x) = -x$, and $(-x^3) = -x^3$

The graph of an even function is symmetric about the y-axis, and also the graph of an odd function is symmetric about the origin, and shown figure below:





Example 4 :

Show which function is even or odd

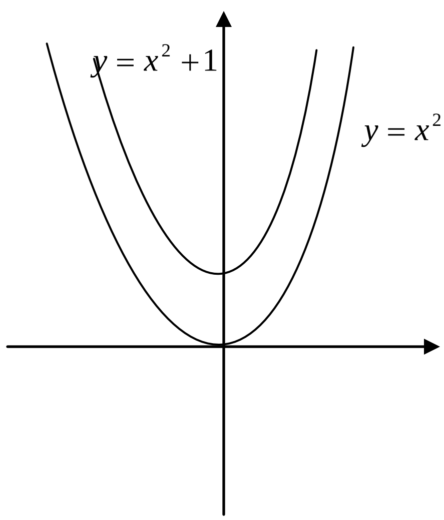
1- $f(x)=x^2$ its even function : $(-x)^2=x^2$ for all x; symmetry about y-axis

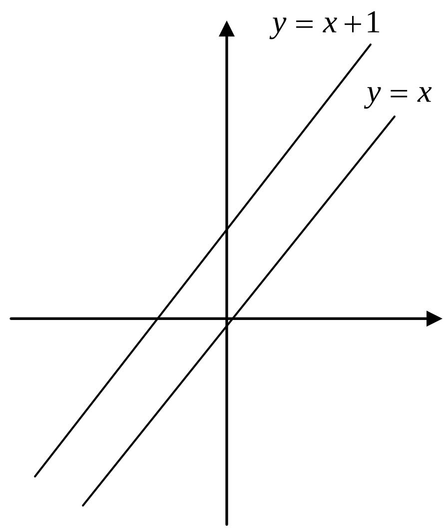
2- $f(x)=x^2+1$ its even function : $(-x)^2+1=x^2+1$ for all x; symmetry about y-axis

3- $f(x)=x$ its odd function : $(-x)=-x$ for all x ; symmetry about the Origin.

4- $f(x)=x+1$ the function is not odd function : $f(-x)=1-x$, but $-f(x)=-1-x$, the two are not equal.

Not even : $(-x)+1\neq x+1$ for all $x\neq 0$



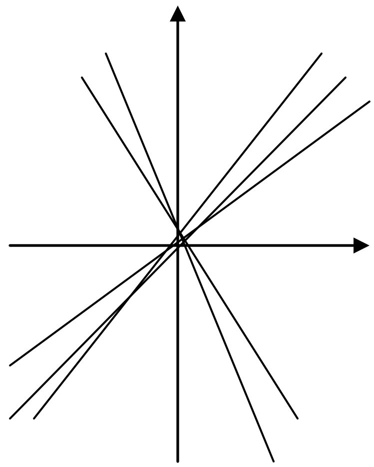
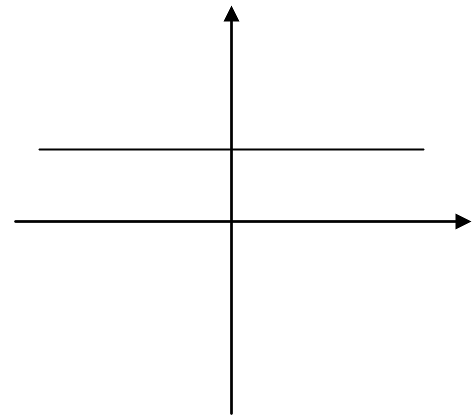


1.5 Common Functions

1- linear functions:

A function of the form $f(x)=mx+b$, where m, b constant

If b=0 $f(x)=mx$ for any value of m, the function shown in graph us array of lines, so these lines pass through the origin.

If m=0 $f(x)=b$ and called constant functions.

A linear function with positive slope (m) whose graph pass through the origin is called proportionality relationship.

<u>Definition</u>

Two variables y and x are proportional (to one another) if one is always a constant multiple of the other; that is, if y=kx for some nonzero constant k.

If the variable y is proportional to the reciprocal 1/x, then sometimes it is said that y is inversely proportional to x (because 1/x is the multiplicative inverse of x).

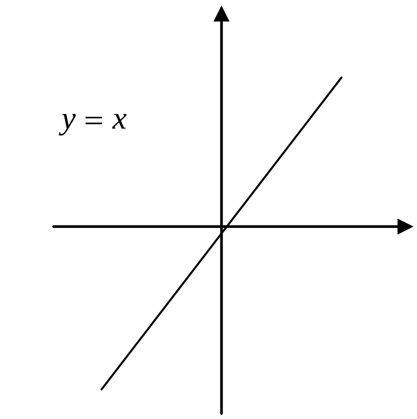
2- power functions:

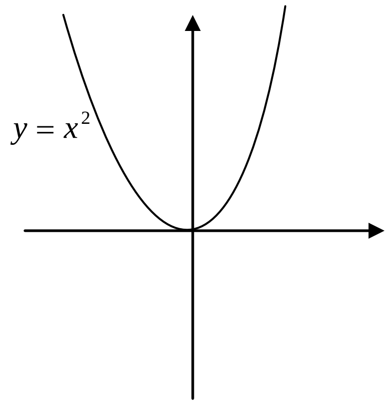
A function $f(x)=x^a$, where a is a constant, is called a power function. There are several important cases to consider.

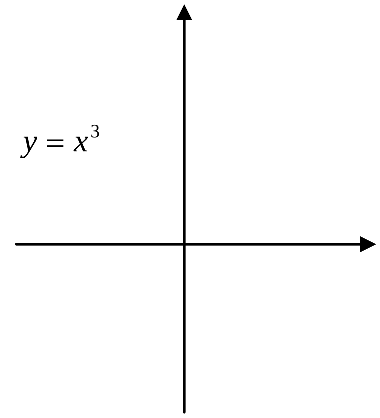
a- $a=n$, n positive integer

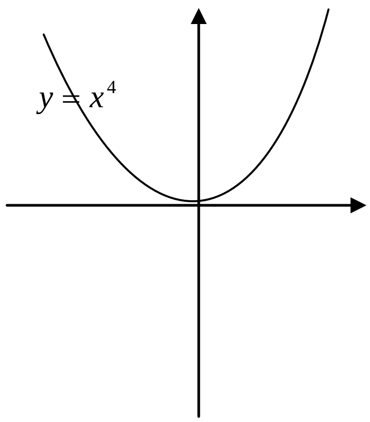
the graphs of $f(x)=x^n$, for $n=1,2,3,4$, are displayed in figure below:

and x be defined $-\infty \langle x \langle \infty$



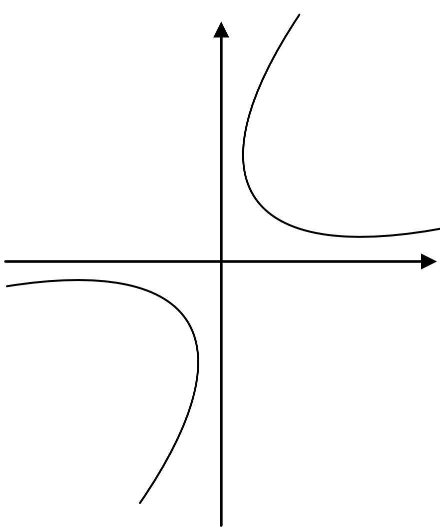


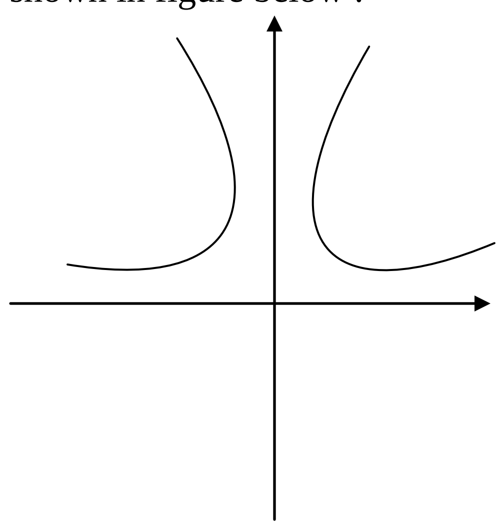




b- $a = n$, n negative integer

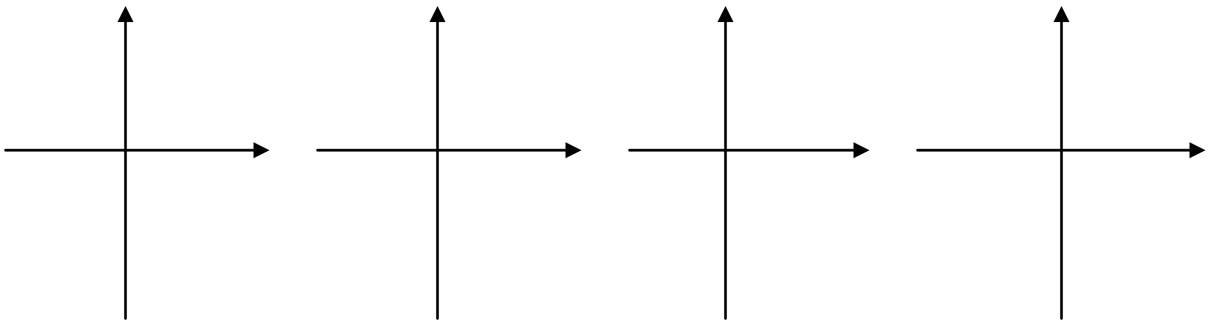
the graphs of the functions, $f(x) = x^{-1} = \frac{1}{x}$, $f(x) = x^{-2} = \frac{1}{x^2}$ where $x \neq 0$, both functions are defined for all are shown in figure below :





c- $a = \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{3}{2}, and\ \frac{2}{3}$

the functions $f(x) = x^{1/2} = \sqrt{x}$ and $g(x) = x^{1/3} = \sqrt[3]{x}$ are the square root and cube root functions, the domain of the square root function is $[0, \infty)$, but the cube root function is defined for all real x, and the graph shown below:



3- polynomials functions

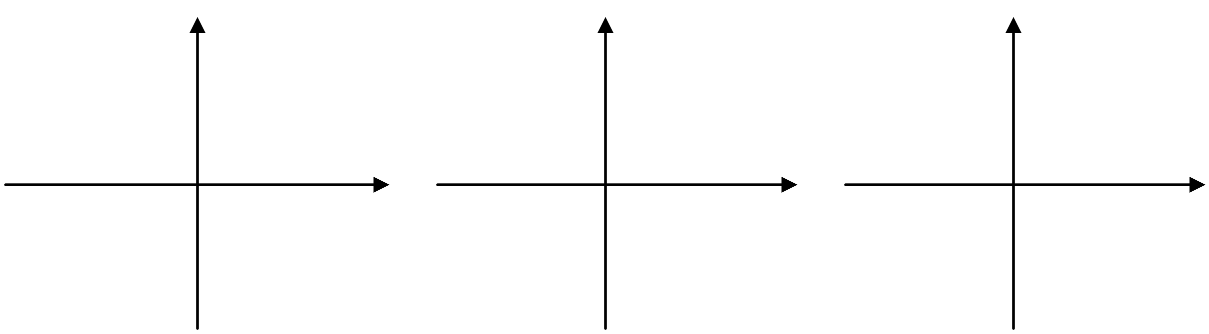
A function p is a polynomial if

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots\dots\dots\dots + a_1 x + a_0$$

Where n is a non-negative integer and the numbers $a_n, a_{n-1}, a_{n-2}, \dots\dots\dots, a_1, a_0$ are real constants (called the coefficients of the polynomial), all polynomials have domain $(-\infty, \infty)$, and n is the degree of polynomial when

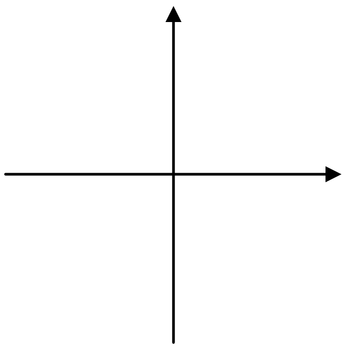
n=2 $p(x) = ax^2 + bx + c$ called quadratic function's

n=3 $p(x) = ax^3 + bx^2 + cx + d$ called cubic function's , and shown in figures below :

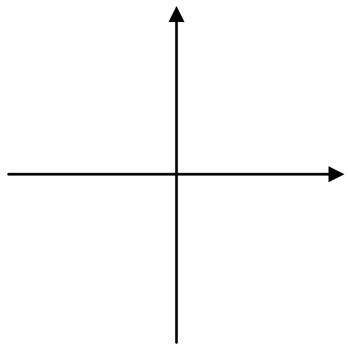


4- Rational functions

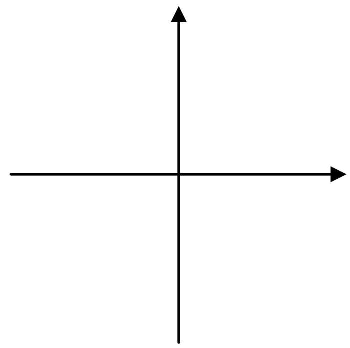
A rational function is a ratio $f(x) = h(x)/g(x)$, where h and g are polynomials. The domain of rational function is the set of all real x for which $g(x) \neq 0$ and the graphs shown as:



$y = \dfrac{(2x^2 - 3)}{(7x + 4)}$



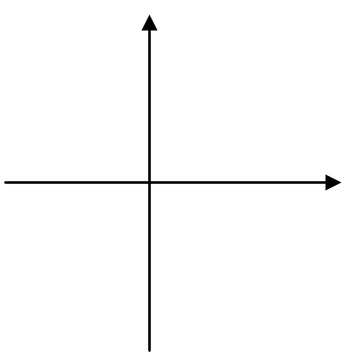
$y = \dfrac{(5x^2 + 8x - 3)}{(3x^2 + 2)}$



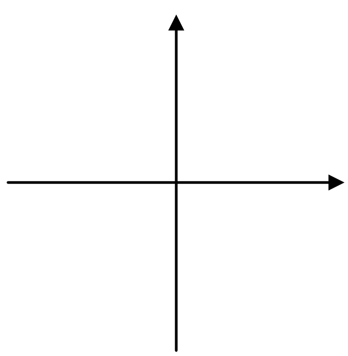
$y = \dfrac{(11x + 2)}{(2x^3 - 1)}$

5- Algebraic functions

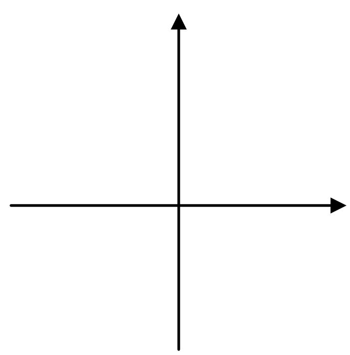
Any function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots). All rational functions are algebraic, but also included are more complicated functions such as $y^3 - 9xy + x^3 = 0$ displays the graphs of there algebraic functions.



$y = x^{1/3}(x - 4)$



$y = \dfrac{3}{4}(x^2 - 1)^{2/3}$



$y = x(1 - x)^{2/5}$

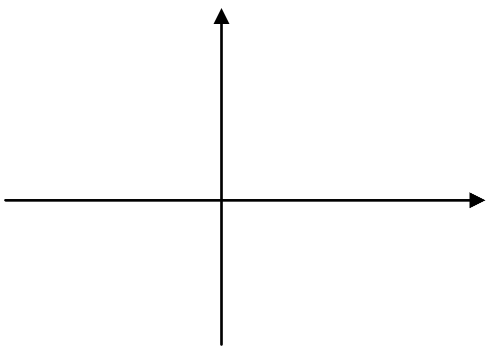
6- Trigonometric functions

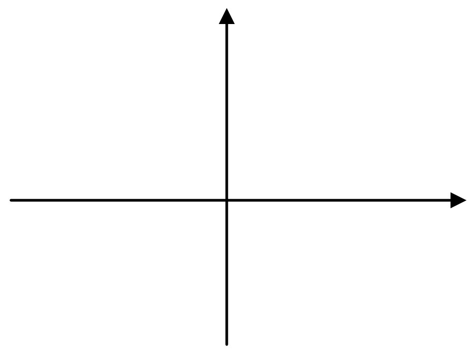
There are six basic trigonometric functions

$f(x)=\sin(x)$ $\qquad$ $f(x)=\csc(x)=\dfrac{1}{\sin(x)}$

$f(x)=\cos(x)$ $\qquad$ $f(x)=\sec(x)=\dfrac{1}{\cos(x)}$

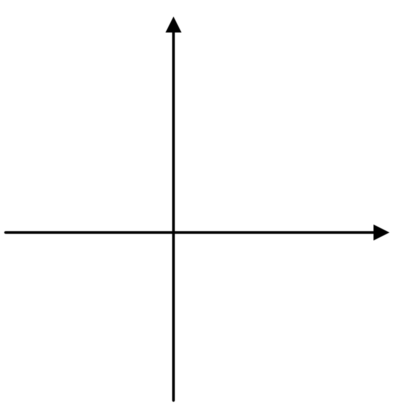
$f(x)=\tan(x)$ $\qquad$ $f(x)=\cot(x)=\dfrac{1}{\tan(x)}$

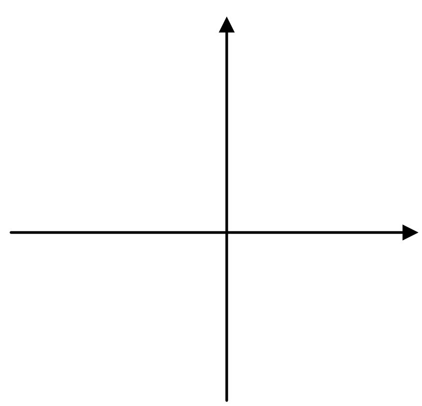




7- Exponential functions

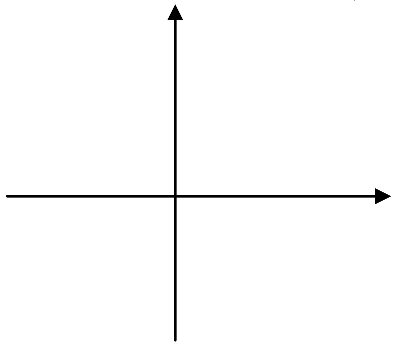
Functions of the form $f(x)=a^x$, where the base $a>0$ is a positive constant and $a\neq 1$, all exp. Functions have domain $(-\infty,\infty)$ and range $(0,\infty)$ and the graph as



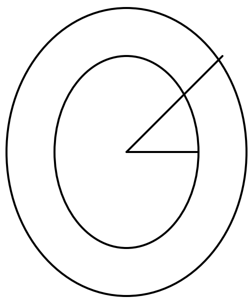


8- Logarithmic functions

These are the functions $f(x)=\log_a x$, where the base $a \neq 1$ is a positive constant. They are the inverse function of the exponential functions, Functions have domain $(0,\infty)$ and range $(-\infty,\infty)$ and the graph as



<u>1.6 Trigonometric Functions</u> :



Angles : angles are measured in degree or radians

$\theta=\dfrac{s}{r} \Rightarrow s=r\theta$ (θ in radians)

If the circle is a unit circle having

Radius $r=1$ show in fig. above , since on complete

Revolution of the unit circle is 360^0 or 2π radians, we have

$\pi \ radians = 180^0$

and

$1 \ radian = \dfrac{180^0}{\pi} (\approx 57.3) \ \deg ree$

or

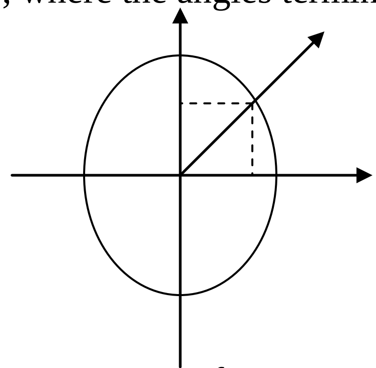
$1 \ \deg ree = \dfrac{\pi}{180^0} (\approx 0.017) \ radians$

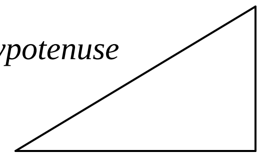
The table above shows the equivalence between degree and radian measures for some basic angles.

degree	-180	-135	-90	-45	0	30	45	60	90	180	360
radian	$-\pi$	$-3\pi/4$	$-\pi/2$	$-\pi/4$	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π	2π

The Six Basic Trigonometric Functions

We define the trigonometric functions in terms of the coordinates of the point $P(x, y)$, where the angles terminal ray intersects the circle as shown as:





The trigonometric ratios of an acute angles are

$$\sin\theta = \frac{opp}{hyp} \qquad \csc\theta = \frac{hyp}{opp}$$

$$\cos\theta = \frac{adj}{hyp} \qquad \sec\theta = \frac{hyp}{adj}$$

$$\tan\theta = \frac{opp}{adj} \qquad \cot\theta = \frac{adj}{opp}$$

The trigonometric functions of a general angles θ are defined in terms of $x, y, and\ r$.

Sine : $\sin\theta = \frac{y}{r}$ cosecant : $\csc\theta = \frac{r}{y}$

Cosine : $\cos\theta = \frac{x}{r}$ secant : $\sec\theta = \frac{r}{x}$

Tangent : $\tan\theta = \frac{y}{x}$ cotangent : $\cot\theta = \frac{x}{y}$

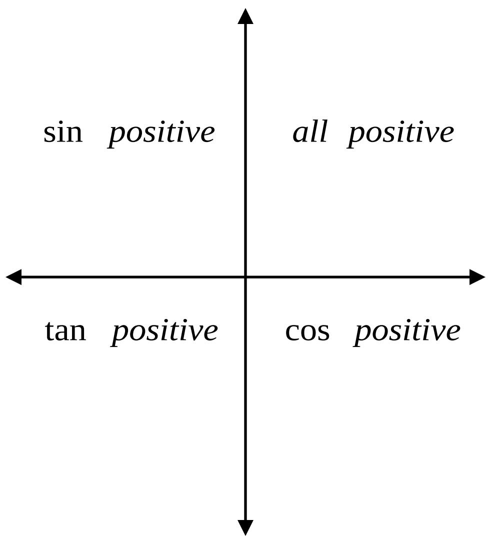
Notice also that whenever the quotients are defined,

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

As you can see, $\tan\theta$ *and* $\sec\theta$ are not defined if $x = \cos\theta = 0$. This means they are not defined if $\theta = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{odd\pi}{2}$, and similarly $\cot\theta$ *and* $\csc\theta$ are not defined for values of θ for which y=0, namely $\theta = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \pm n\pi$

The CAST rule is useful for remembering when the basic trigonometric functions are positive or negative, see figure below:



Using a similar method we determined the values of $\sin\theta$, $\cos\theta$ *and* $\tan\theta$ shown as:

degree	-180	-135	-90	-45	0	30	45	60	90	120	135	150	180	270	360
θ(rad)	$-\pi$	$-\frac{3\pi}{4}$	$-\frac{\pi}{2}$	$-\frac{\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$-\frac{1}{\sqrt{2}}$	-1	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	-1	0
$\cos\theta$	-1	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	0	1
$\tan\theta$	0	1	∞	-1	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	∞	0

Periodicity and Graphs of the Trigonometric functions:

When an angle of measure θ and an angle of measure $\theta + 2\pi$ are standard position, their terminal rays coincide. The two angles therefore have the same trigonometric function values:

$\sin(\theta + 2\pi) = \sin\theta$, $\tan(\theta + 2\pi) = \tan\theta$, $\cos(\theta - 2\pi) = \cos\theta$, $\sin(\theta - 2\pi) = \sin\theta$, we describe this repeating behavior by saying that the six basic trigonometric functions are periodic.

Definition:

A function $f(x)$ is periodic if there is a positive number p such that $f(x + p) = f(x)$ for every values of x.

Periods of trigonometric functions:

1- period (π): $\tan(x + \pi) = \tan x$

$\cot(x + \pi) = \cot x$

2- period (2π): $\sin(x + 2\pi) = \sin x$

$\cos(x + 2\pi) = \cos x$

$\sec(x + 2\pi) = \sec x$

$\csc(x + 2\pi) = \csc x$

Even functions

$\cos(-x) = \cos x$

$\sec(-x) = \sec x$

Odd functions

$\sin(-x) = -\sin x$

$\tan(-x) = -\tan x$

$\csc(-x) = -\csc x$

$\cot(-x) = -\cot x$

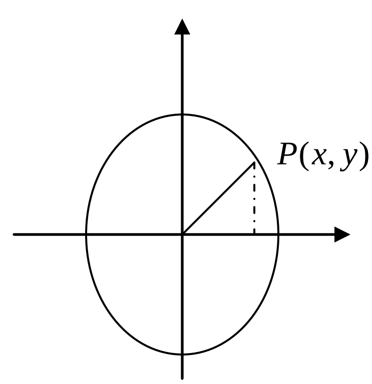
The graph of trigonometric functions in the coordinate plane shown as:

Trigonometric Identities

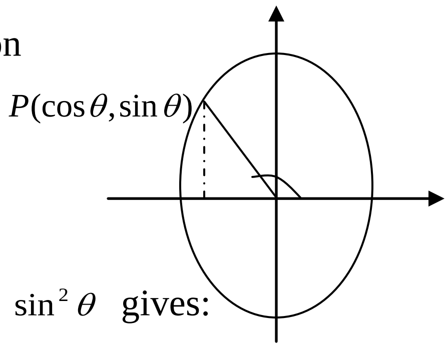
The coordinates of any point $P(x, y)$ in the plane can be expressed in terms of the point's distance r from the origin and the angle θ that ray OP makes with the positive x-axis.

Since $\frac{x}{r} = \cos\theta$ and $\frac{y}{r} = \sin\theta$

Also $x = r\cos\theta$, $y = r\sin\theta$



When $r = 1$, we can apply the Pythagorean theorem to the reference right triangle in fig. below, and obtain the equation



$\cos^2\theta + \sin^2\theta = 1$

Dividing this identity in turn by $\cos^2\theta$ and $\sin^2\theta$ gives:

$1 + \tan^2\theta = \sec^2\theta$

$1 + \cot^2\theta = \csc^2\theta$

The following formulas hold for all angles A and B

Addition Formulas

$\cos(A + B) = \cos A \cos B - \sin A \sin B$

$\sin(A + B) = \sin A \cos B + \cos A \sin B$

Subtraction Formulas

$\cos(A - B) = \cos A \cos B + \sin A \sin B$

$\sin(A - B) = \sin A \cos B - \cos A \sin B$

Double-angle Formulas

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2 x$

$\sin 2\theta = 2\sin\theta\cos\theta$

Half-angle Formulas

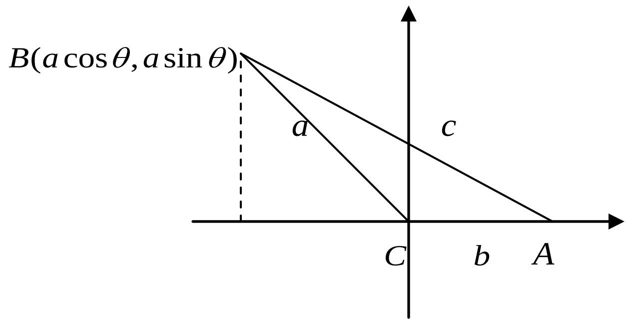
$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

The Law of cosines

If a,b and c are sides of a triangle ABC and if θ is the angle opposite c, then

$c^2 = a^2 + b^2 - 2ab\cos\theta$



The coordinates of A are (b,0)

The coordinates of B are ($a\cos\theta$, $a\sin\theta$)

The coordinates of C are (0,0)

The square of the distance between A and B is therefore:

$c^2 = (a\cos\theta - b)^2 + (a\sin\theta)^2$

$c^2 = a^2(\cos^2\theta + \sin^2\theta) + b^2 - 2ab\cos\theta$

$c^2 = a^2 + b^2 - 2ab\cos\theta$

The Law of cosines generalizes the Pythagorean theorem if $\theta = \pi/2$, then $\cos\theta = 0$ and $c^2 = a^2 + b^2$

Two special Inequalities

For any angles θ measured in radians,

$-|\theta| \le \sin\theta \le |\theta|$ and $-|\theta| \le 1 - \cos\theta \le |\theta|$