3.1 Tangents and the derivative at a point

To find a tangent to an arbitrary curve $y = f(x)$ at a point $P(x_0, f(x_0))$, we calculate the slope of the secant through P and a nearby point $Q(x_0 + h, f(x_0 + h))$, we then investigate the limit of the slope as $h \to 0$. If the limit exists, we call it the slope of the curve at P and define the tangent at P to be the line through P having this slope.

Definition

The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h\to 0}\frac{f(x_0 + h) - f(x_0)}{h}$$ (provided the limit exists)

The **tangent line** to the curve at P is the line through P with this slope.

Example 1 :

Find the slope of the curve $y = 1/x$ at any point $x = a \neq 0$.

1- what is the slope at the point $x = -1$?

2- Where dose the slope equal $-1/4$?

Solution

1- here $f(x) = 1/x$. The slope at (a, 1/a) is

$$\lim_{h\to 0}\frac{f(a+h) - f(a)}{h} = \lim_{h\to 0}\frac{\frac{1}{a+h} - \frac{1}{a}}{h} = \lim_{h\to 0}\frac{1}{h}\frac{a-(a+h)}{a(a+h)}$$

$$= \lim_{h\to 0}\frac{-h}{ha(a+h)} = \lim_{h\to 0}\frac{-1}{a(a+h)} = -\frac{1}{a^2}$$

When $x = -1$, the slope is $\frac{-1}{(-1)^2} = -1$

2- the slope of $y = 1/x$ at the point where $-\frac{1}{a^2}$ is $-\frac{1}{a^2}$ .it will be $-\frac{1}{4}$ provided that

$$-\frac{1}{a^2} = -\frac{1}{4}$$

This equation is equivalent to $a^2 = 4$, so $a = 2$ or $a = -2$. the curve has slope $-\frac{1}{4}$ at the two points $\left(2, \frac{1}{2}\right)$ and $\left(-2, -\frac{1}{2}\right)$.

Definition

The **derivative of a function** f **at** a point x_0, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Provided this limit exists.

3.2 The derivative as a function

Definition The derivative of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Where $h = \Delta x = x_2 - x$

Provided the limit exists.

If we write $z = h + x$, then $h = z - x$ and h approaches 0 if and only if z approaches x. Therefore, an equivalent definition of the derivative is as follows. This formula is sometimes more convenient to use when finding a derivative function.

Alternative formula for the derivative

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

and some common alternative notations for the derivative are

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x)$$

Example 2 :

Differentiate $f(x)=\frac{x}{x-1}$

Solution

We use the definition of the derivative $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$f(x)=\frac{x}{x-1}$ and $f(x+h)=\frac{(x+h)}{(x+h)-1}$, so

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{\frac{(x+h)}{(x+h-1)}-\frac{x}{x-1}}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\cdot\frac{(x+h)(x-1)-x(x+h-1)}{(x+h-1)(x-1)} \qquad \left[\frac{a}{b}-\frac{c}{d}=\frac{ad-cb}{bd}\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\cdot\frac{-h}{(x+h-1)(x-1)}$$

$$=\lim_{h\to 0}\frac{-1}{(x+h-1)(x-1)}=\frac{-1}{(x-1)^2}$$

Example 3 :

a- find the derivative of $f(x)=\sqrt{x}$ for $x>0$.

b- find the tangent line to the curve $y=\sqrt{x}$ at $x=4$.

Solution

a- we use the alternative formula to calculate f':

$$f'(x)=\lim_{z\to x}\frac{f(z)-f(x)}{z-x}$$

$$=\lim_{z\to x}\frac{\sqrt{z}-\sqrt{x}}{z-x}$$

$$=\lim_{z\to x}\frac{\left(\sqrt{z}-\sqrt{x}\right)}{\left(\sqrt{z}-\sqrt{x}\right)\cdot\left(\sqrt{z}+\sqrt{x}\right)}$$

$$=\lim_{z\to x}\frac{1}{\left(\sqrt{z}+\sqrt{x}\right)}=\frac{1}{2\sqrt{x}}$$

b- the slope of the curve at $x = 4$ is

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$$

The tangent is the line through the point $(4,2)$ with slope $(1/4)$

$$y = 2 + \frac{1}{4}(x-4)$$
$$y = \frac{1}{4}x + 1$$

Example 4 :

Show that the function $y = |x|$ is differentiable on $(-\infty,0)$ and $(0,\infty)$ but has no derivative at $x = 0$.

Solution

The derivative of function $y = |x|$ to the right of the origin (positive x)

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(x) = \frac{d}{dx}(1\cdot x) = 1$$

And to the left of the origin (negative x)

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = \frac{d}{dx}(-1\cdot x) = -1$$

There is no derivative at the origin because the one-sided derivatives differ there:

Right-hand derivative of $|x|$ at zero $= \lim_{h\to 0^+} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^+} \frac{|h|}{h}$

$$= \lim_{h\to 0^+} \frac{h}{h}$$
$$= \lim_{h\to 0^+} (1) = 1$$

left-hand derivative of $|x|$ at zero $= \lim_{h\to 0^-} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^-} \frac{|h|}{h}$

$$= \lim_{h\to 0^-} \frac{-h}{h}$$
$$= \lim_{h\to 0^-} (-1) = -1$$

3.3 Differentiation Rules

Powers, Multiples, Sums, and Differences

A simple rule of differentiation is that the derivative of every constant function is zero.

1- Derivative of a constant function

if f has the constant value $f(x)=c$, then

$$\frac{df}{dx}=\frac{d}{dx}(c)=0$$

Proof we apply the definition of the derivative to $f(x)=c$, the function whose outputs have the constant value c, at every value of x, we find that

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{c-c}{h}=\lim_{h\to 0}(0)=0$$

2- Power Rule for positive integers:

if n is a positive integer, then

$$\frac{df}{dx}x^{n}=n\cdot x^{n-1}$$

Proof the formula

$$z^{n}-x^{n}=(z-x)\cdot\left(z^{n-1}+z^{n-2}x+........+zx^{n-2}+x^{n-1}\right)$$

From the alternative formula for the definition of the derivative,

$$\begin{aligned} f'(x)&=\lim_{z\to x}\frac{f(z)-f(x)}{z-x}=\lim_{z\to x}\frac{z^{n}-x^{n}}{z-x}\\ &=\lim_{z\to x}\frac{(z-x)\cdot\left(z^{n-1}+z^{n-2}x+........+zx^{n-2}+x^{n-1}\right)}{(z-x)}\\ &=n\cdot x^{n-1}\end{aligned}$$

The power rule is actually valid for all real numbers n (positive or negative).

For ex. $\frac{d}{dx}\left(\frac{1}{x}\right)=\frac{d}{dx}\left(x^{-1}\right)=-x^{-2}$

$$\frac{d}{dx}\left(\sqrt{x}\right)=\frac{d}{dx}\left(x^{1/2}\right)=\frac{1}{2}x^{-1/2}$$

<u>Example 5 :</u>

Differentiate the following powers of x .

(a) x^3 (b) $x^{2/3}$ (c) $x^{\sqrt{2}}$ (d) $\frac{1}{x^4}$ (e) $x^{-4/3}$ (f) $\sqrt{x^{2+\pi}}$

<u>Solution</u>

(a) $\frac{d}{dx}\left(x^3\right)=3\cdot x^{3-1}=3x^2$ (b) $\frac{d}{dx}\left(x^{2/3}\right)=\left(2/3\right)\cdot x^{(2/3)-1}=\left(2/3\right)\cdot x^{-1/3}$

(c) $\frac{d}{dx}\left(x^{\sqrt{2}}\right)=\sqrt{2}\cdot x^{\sqrt{2}-1}$ (d) $\frac{d}{dx}\left(\frac{1}{x^4}\right)=\frac{d}{dx}\left(x^{-4}\right)=-4\cdot\left(x^{-4-1}\right)=-4\cdot x^{-5}=-\frac{4}{x^5}$

(e) $\frac{d}{dx}\left(x^{-4/3}\right)=-\left(4/3\right)\cdot x^{-(4/3)-1}=-\left(4/3\right)\cdot x^{-(7/3)}$

(f) $\frac{d}{dx}\left(\sqrt{x^{2+\pi}}\right)=\frac{d}{dx}\left(x^{1+(\pi/2)}\right)=\left(1+\frac{\pi}{2}\right)\cdot\left(x^{1+(\pi/2)-1}\right)=\left(\frac{1}{2}\right)\cdot\left(2+\pi\right)\cdot\left(x^{(\pi/2)}\right)=\left(\frac{1}{2}\right)\cdot\left(2+\pi\right)\cdot\left(\sqrt{x^{\pi}}\right)$

3- Derivative constant multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu)=c\frac{du}{dx}$$

In particular, if n is any real number, then

$$\frac{d}{dx}\left(cx^n\right)=\left(c\cdot n\right)\cdot x^{n-1}$$

Proof

$$\begin{aligned}\frac{d}{dx}(cu)&=\lim_{h\to 0}\frac{cu(x+h)-cu(x)}{h}\\&=c\cdot\lim_{h\to 0}\frac{u(x+h)-u(x)}{h}\\&=c\cdot\frac{du}{dx}\end{aligned}$$

<u>Example 6 :</u>

(a) the derivative formula $3x^2$ is

$$\frac{d}{dx}\left(3x^2\right)=(3\cdot 2)\cdot x=6x$$

We describe that, the graph of $y=x^2$ multiplying y-coordinate by 3, then we multiply the slope at each point by 3.

(b) negative of a function

The derivative of the negative of a differentiable function u is the negative of the function's derivative. The constant multiple Rule with $c=-1$ gives

$$\frac{d}{dx}(-u)=\frac{d}{dx}(-1\cdot u)=-1\cdot\frac{d}{dx}u=-\frac{du}{dx}$$

4- Derivative Sum Rule

If u and v are differentiable function of x, then their Sum $u+v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u+v)=\frac{du}{dx}+\frac{dv}{dx}$$

For ex., if $y=x^4+12x$, then y is the sum of $u(x)=x^4$ and $v(x)=12x$. We then have

$$\frac{d}{dx}y=\frac{d}{dx}\left(x^4\right)+\frac{d}{dx}(12x)=4x^3+12$$

<u>Example 7 :</u>

Find the derivative of the polynomial $y=x^3+(4/3)x^2-5x+1$

Solution

$$\frac{dy}{dx}=\frac{d}{dx}\left(x^3\right)+\frac{d}{dx}\left((4/3)x\right)-\frac{d}{dx}(5x)+\frac{d}{dx}(1)$$

$$\frac{dy}{dx}=3x^2+(4/3)\cdot(2x)-5+0=3x^2+(8/3)\cdot x-5$$

5- Derivative Product Rule

If u and v are differentiable function at x, then so is their product uv, and

$$\frac{d}{dx}(u\cdot v)=u\frac{dv}{dx}+v\frac{du}{dx} \quad \text{, in prime notation, } (u\cdot v)'=uv'+vu'$$

Example 8 :

Find the derivative of $y=\left(x^2+1\right)\cdot\left(x^3+3\right)$

Solution

From the product Rule with $u=\left(x^2+1\right)$ and $v=\left(x^3+3\right)$, we find

$$\begin{aligned}\frac{d}{dx}\left(x^2+1\right)\cdot\left(x^3+3\right)&=\left(x^2+1\right)\frac{d}{dx}\left(x^3+3\right)+\left(x^3+3\right)\frac{d}{dx}\left(x^2+1\right)\\&=\left(x^2+1\right)\cdot\left(3x^2\right)+\left(x^3+3\right)\cdot(2x)\\&=3x^4+3x^2+2x^4+6x\\&=5x^4+3x^2+6x\end{aligned}$$

6- Derivative Quotient Rule

If u and v are differentiable function at x and if $v(x)\neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$$

Example 9 :

Find the derivative of $y=\frac{\left(t^2-1\right)}{\left(t^3+1\right)}$

<u>Solution</u>

$$\frac{dy}{dx}=\frac{\left(t^3+1\right)\frac{d}{dx}\left(t^2-1\right)-\left(t^2-1\right)\frac{d}{dx}\left(t^3+1\right)}{\left(t^3+1\right)^2}$$

$$=\frac{\left(t^3+1\right)\cdot(2t)-\left(t^2-1\right)\cdot\left(3t^2\right)}{\left(t^3+1\right)^2}$$

$$=\frac{\left(2t^4+2t-3t^4+3t^2\right)}{\left(t^3+1\right)^2}$$

$$=\frac{\left(-t^4+3t^2+2t\right)}{\left(t^3+1\right)^2}$$

7- Second- and Higher-order Derivatives

If $y=f(x)$ is the differentiable function, then its derivative $f'(x)$ is also a function. If f' is also differentiable, then we can differentiate f' to get a new function of x denoted by f''. The function f'' is called the **second derivative** of f and written in several ways:

$$f''(x)=y''=\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)$$

If y'' is differentiable, its derivative, $y'''=dy''/dx=d^3y/dx^3$, is the **third derivative** of y with respect to x. also for ***nth derivative*** of y with respect to x for any positive integer n, given as

$$y^{(n)}=\frac{d}{dx}y^{(n-1)}=\left(\frac{d^ny}{dx^n}\right)$$

<u>Example 10 :</u>

Find the first four derivatives of $y=x^3-3x^2+2$ are

first derivative: $y'=3x^2-6x$

second derivative: $y''=6x-6$

third derivative: $y'''=6$

fourth derivative: $y^{(4)}=0$

3.4 Derivatives of Trigonometric Functions

1. Derivative of the Sine Function

To calculate the derivative of $f(x)=\sin(x)$, for x measured in radians, we combine the limits

$$\sin(x+h)=\sin(x)\cos(h)+\cos(x)\sin(h)$$

If $f(x)=\sin(x)$, then

$$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to0}\frac{\sin(x+h)-\sin(x)}{h}$$

$$=\lim_{h\to0}\frac{(\sin(x)\cos(h)+\cos(x)\sin(h))-\sin(x)}{h}=\lim_{h\to0}\frac{\sin(x)\cdot(\cos(h)-1)+(\cos(x)\sin(h))}{h}$$

$$=\lim_{h\to0}\left(\sin(x)\cdot\frac{(\cos(h)-1)}{h}\right)+\lim_{h\to0}\left(\cos(x)\cdot\frac{\sin(h)}{h}\right)$$

$$=\sin(x)\cdot\lim_{h\to0}\left(\frac{(\cos(h)-1)}{h}\right)+\cos(x)\cdot\lim_{h\to0}\left(\frac{\sin(h)}{h}\right)=\sin(x)\cdot(0)+\cos(x)\cdot(1)=\cos(x)$$

Since, the derivative of the sine function is the cosine function:

$$\frac{d}{dx}\sin(x)=\cos(x)$$

Example 11 :

Find the derivatives of the sine function involving differences, products, and quotients: (a) $y=x^2-\sin(x)$ (b) $y=x^2\cdot\sin(x)$ (c) $y=\frac{\sin(x)}{x}$

Solution

(a) $y=x^2-\sin(x)$: $\frac{dy}{dx}=2x-\frac{d}{dx}\sin(x)=2x-\cos(x)$

(b) $y=x^2\cdot\sin(x)$: $\frac{dy}{dx}=x^2\cdot\frac{d}{dx}\sin(x)+2x\cdot\sin(x)=x^2\cdot\cos(x)+2x\cdot\sin(x)$

(c) $y=\frac{\sin(x)}{x}$: $\frac{dy}{dx}=\frac{x\cdot\frac{d}{dx}\sin(x)-\sin(x)\cdot(1)}{x^2}=\frac{x\cdot\cos(x)-\sin(x)}{x^2}$

2. Derivative of the Cosine Function

To calculate the derivative of $f(x)=\cos(x)$, for x measured in radians, we combine the limits

$\cos(x+h)=\cos(x)\cos(h)-\sin(x)\sin(h)$

If $f(x)=\cos(x)$, then we can compute the limit of the difference quotient:

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\cos(x+h)-\cos(x)}{h}$$
$$=\lim_{h\to 0}\frac{(\cos(x)\cos(h)-\sin(x)\sin(h))-\cos(x)}{h}=\lim_{h\to 0}\frac{\cos(x)\cdot(\cos(h)-1)-(\sin(x)\sin(h))}{h}$$
$$=\lim_{h\to 0}\left(\cos(x)\cdot\frac{(\cos(h)-1)}{h}\right)-\lim_{h\to 0}\left(\sin(x)\cdot\frac{\sin(h)}{h}\right)$$
$$=\cos(x)\cdot\lim_{h\to 0}\left(\frac{(\cos(h)-1)}{h}\right)-\sin(x)\cdot\lim_{h\to 0}\left(\frac{\sin(h)}{h}\right)=\cos(x)\cdot(0)-\sin(x)\cdot(1)=-\sin(x)$$

Since, the derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}\cos(x)=-\sin(x)$$

Example 12 :

Find the derivatives of the cosine function in combinations with other functions: (a) $y=5x+\cos(x)$ (b) $y=\sin(x)\cdot\cos(x)$ (c) $y=\dfrac{\cos(x)}{1-\sin(x)}$

Solution

(a) $y=5x+\cos(x)$: $\dfrac{dy}{dx}=\dfrac{d}{dx}(5x)+\dfrac{d}{dx}\cos(x)=5-\sin(x)$

(b) $y=\sin(x)\cdot\cos(x)$: $\dfrac{dy}{dx}=\sin(x)\cdot\dfrac{d}{dx}(\cos(x))+\cos(x)\cdot\dfrac{d}{dx}(\sin(x))$

$$=\sin(x)\cdot(-\sin(x))+\cos(x)\cdot\cos(x)=\cos^2(x)-\sin^2(x)$$

(c) $y=\dfrac{\cos(x)}{1-\sin(x)}$:

$$\frac{dy}{dx}=\frac{(1-\sin(x))\cdot\frac{d}{dx}\cos(x)-\cos(x)\cdot\frac{d}{dx}(1-\sin(x))}{(1-\sin(x))^2}$$

$$=\frac{(1-\sin(x))\cdot(-\sin(x))-\cos(x)\cdot(0-\cos(x))}{(1-\sin(x))^2}$$

$$=\frac{(-\sin(x)+\sin^2(x))+\cos^2(x)}{(1-\sin(x))^2}$$

$$=\frac{(1-\sin(x))}{(1-\sin(x))^2}=\frac{1}{(1-\sin(x))} \qquad \left[\cos^2(x)+\sin^2(x)=1\right]$$

3. Derivatives of the other Basic Trigonometric Functions

Because $\sin(x)$ and $\cos(x)$ are differentiable functions of x, the related functions $\tan(x)=\dfrac{\sin(x)}{\cos(x)}$, $\cot(x)=\dfrac{\cos(x)}{\sin(x)}$, $\sec(x)=\dfrac{1}{\cos(x)}$, $\csc(x)=\dfrac{1}{\sin(x)}$, are differentiable at every value of x at which they are defined. Their derivatives, calculated from the Quotient Rule, are given by the following formulas:

$$\frac{d}{dx}\tan(x)=\sec^2(x) \qquad \frac{d}{dx}\cot(x)=-\csc^2(x)$$

$$\frac{d}{dx}\sec(x)=\sec(x)\cdot\tan(x) \qquad \frac{d}{dx}\csc(x)=-\csc(x)\cdot\cot(x)$$

Example 13 : Find $\dfrac{d}{dx}\tan(x)$

Solution

$$\frac{d}{dx}\tan(x)=\frac{d}{dx}\left(\frac{\sin(x)}{\cos(x)}\right)=\left(\frac{\cos(x)\cdot\frac{d}{dx}\sin(x)-\sin(x)\cdot\frac{d}{dx}\cos(x)}{\cos^2(x)}\right)$$

$$=\left(\frac{\cos(x)\cdot\cos(x)-\sin(x)\cdot(-\sin(x))}{\cos^2(x)}\right)$$

$$=\frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)}=\frac{1}{\cos^2(x)}=\sec^2(x)$$

Example 14 : Find y'' if $y = \sec(x)$

Solution

$y' = \sec(x)\cdot \tan(x)$

$$y'' = \frac{d}{dx}(\sec(x)\cdot \tan(x))$$
$$= \sec(x)\cdot \frac{d}{dx}\tan(x) + \tan(x)\cdot \frac{d}{dx}\sec(x)$$
$$= \sec(x)\cdot \sec^2(x) + \tan(x)\cdot (\sec(x)\tan(x))$$
$$= \sec^3(x) + \tan^2(x)\cdot \sec(x)$$
$$= \sec(x)\cdot (\sec^2(x) + \tan^2(x))$$

3.5 The chain Rule

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x))\cdot g'(x)$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du}\cdot \frac{du}{dx}$$

Where $\frac{dy}{du}$ is evaluated at $u = g(x)$.

Example 15 : Find the derivative of the function $y = (3x^2 + 1)^2$

Solution

The function here is composite of $y = f(u) = u^2$ and $u = g(x) = 3x^2 + 1$,

therefore, $\frac{dy}{du} = f'(u) = 2u$ and $\frac{du}{dx} = g'(x) = 6x$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot \frac{du}{dx} = 2u \cdot 6x = 2 \cdot (3x^2 + 1)\cdot 6x \qquad = 36x^2 + 12x$$

Example 16 : An object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t)=\cos(t^2+1)^2$. Find the velocity of the object as a function of t.

Solution

We know that the velocity is $\frac{dx}{dt}$. In this instance, x is a composite function:

$x=\cos(u)$ and $u=t^2+1$, we have

$\frac{dx}{du}=-\sin(u)$ and $\frac{du}{dt}=2t$

By the chain Rule,

$$\frac{dx}{dt}=\frac{dx}{du}\cdot\frac{du}{dt}=-\sin(u)\cdot 2t=-\sin(t^2+1)\cdot 2t=-2t\sin(t^2+1)$$

Example 17 : Differentiate $\sin(x^2+x)$ with respect to x.

Solution We apply the Chain Rule directly and find

$$\frac{d}{dx}\sin(x^2+x)=\cos(x^2+x)\cdot(2x+1)$$

Example 18 : Find the derivative of the function $g(t)=\tan(5-\sin(2t))$

Solution

$$g'(t)=\frac{d}{dt}[\tan(5-\sin(2t))]$$
$$=\sec^2(5-\sin(2t))\cdot\frac{d}{dt}(5-\sin(2t))$$
$$=\sec^2(5-\sin(2t))\cdot\left(0-\cos(2t)\cdot\frac{d}{dt}(2t)\right)$$
$$=\sec^2(5-\sin(2t))\cdot(-\cos(2t))\cdot(2)$$
$$=(-2\cos(2t))\cdot\sec^2(5-\sin(2t))$$

The Chain Rule with Powers of function

If f is a differentiable function of u and if u is a differentiable function of x, then substituting $y = f(u)$ into the Chain Rule formula

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Leads to the formula

$$\frac{d}{dx} f(u) = f'(u)\frac{du}{dx}$$

If n is any real number and f is a power function, $f(u) = u^n$, the power rule tells us that $f'(u) = nu^{n-1}$. if u is a differentiable function of x, then we can use the chain rule to extend this to the **Power Chain Rule**.

$$\frac{d}{dx}\left(u^n\right) = n \cdot u^{n-1}\frac{du}{dx}$$

<u>Example 19 :</u> Find the derivative of a power of an expressions follows:

(a) $\frac{d}{dx}\left(5x^3 - x^4\right)^7$ (b) $\frac{d}{dx}\left(\frac{1}{3x-2}\right)$ (c) $\frac{d}{dx}\left(\sin^5(x)\right)$

<u>Solution</u>

(a)

$$\begin{aligned}\frac{d}{dx}\left(5x^3 - x^4\right)^7 &= 7 \cdot \left(5x^3 - x^4\right)^6 \cdot \frac{d}{dx}\left(5x^3 - x^4\right)\\ &= 7 \cdot \left(5x^3 - x^4\right)^6 \cdot \left(5 \cdot 3x^2 - 4x^3\right)\\ &= 7 \cdot \left(5x^3 - x^4\right)^6 \cdot \left(15x^2 - 4x^3\right)\end{aligned}$$

(b)

$$\begin{aligned}\frac{d}{dx}\left(\frac{1}{3x-2}\right) &= \frac{d}{dx}(3x-2)^{-1}\\ &= -1 \cdot (3x-2)^{-2} \cdot \frac{d}{dx}(3x-2)\\ &= -1 \cdot (3x-2)^{-2} \cdot (3)\\ &= -\frac{3}{(3x-2)^2}\end{aligned}$$

(c) $\frac{d}{dx}\left(\sin^5(x)\right) = 5\cdot\sin^4(x)\cdot\frac{d}{dx}\sin(x)$

$$= 5\cdot\sin^4(x)\cdot\cos(x)$$

Example 20 : show that the slope of every line tangent to the curve $y = \left(\frac{1}{(1-2x)^3}\right)$ is positive.

Solution

we find the derivative:

$$\frac{dy}{dx} = \frac{d}{dx}\left((1-2x)^{-3}\right)$$

$$= \left(-3\cdot(1-2x)^{-4}\right)\cdot\frac{d}{dx}(1-2x)$$

$$= \left(-3\cdot(1-2x)^{-4}\right)\cdot(-2)$$

$$= \frac{6}{(1-2x)^4}$$

At any point (x, y) on the curve, $x \neq 1/2$ and the slope of the tangent line is

$$\frac{dy}{dx} = \frac{6}{(1-2x)^4}$$

The quotient of two positive numbers.

3.6 Implicit Differentiation

Most of the functions described by an equation of the form $y = f(x)$. Another situation occurs when we encounter equations like

$x^3 + y^3 - 9xy = 0$, $\quad y^2 - x = 0$, $\quad x^2 + y^2 - 25 = 0$

To calculate these types of function, we treat (y) as a differentiable implicit function of (x) and apply the usual rules to differentiate both sides of the defining equation.

1. differentiate both sides of the equation with respect to (x), treating (y) as a differentiable function of (x).

2. collect the terms with $\frac{dy}{dx}$ on one side of the equation and solve for $\frac{dy}{dx}$

Example 21 : find $\frac{dy}{dx}$ if $y^2 = x$.

Solution

The equation $y^2 = x$ defines two differentiable functions of x that we can actually find, namely $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$, then

$$\frac{dy_1}{dx} = \frac{1}{2\sqrt{x}}$$

$$\frac{dy_2}{dx} = -\frac{1}{2\sqrt{x}}$$

By other way to find $\frac{dy}{dx}$, we simply differentiable both sides of the equation $y^2 = x$ with respect to x, treating $y = f(x)$ as a differentiable function of x:

$y^2 = x$ the chain rule gives $\frac{d}{dx}(y^2) = \frac{d}{dx}[f(x)]^2 = 2f(x)f'(x) = 2y\frac{dy}{dx}$

$$2y \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{2y}$$

This one formula gives the derivative we calculated for both explicit solutions $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$:

$$\frac{dy_1}{dx} = \frac{1}{2y_1} = \frac{1}{2\sqrt{x}}$$

$$\frac{dy_2}{dx} = \frac{1}{2y_2} = \frac{1}{2(-\sqrt{x})} = -\frac{1}{2\sqrt{x}}$$

Example 22 : find the slope of the circle $x^2 + y^2 = 25$ at the point (3,-4).

Solution

The circle is combined graphs of two differentiable functions, $y_1 = \sqrt{25 - x^2}$ and $y_2 = -\sqrt{25 - x^2}$. The point (3,-4) lies on the graph of y_2, so we can find the slope by calculating the derivative directly, using the power chain rule:

$\left[\frac{d}{dx}\left(-\left(25 - x^2\right)^{1/2}\right)\right] = -\frac{1}{2}\left(25 - x^2\right)^{-1/2} \cdot (-2x)$, then

$$\left.\frac{dy_2}{dx}\right|_{x=3} = -\left.\frac{-2x}{2\sqrt{25 - x^2}}\right|_{x=3} = -\frac{-6}{2\sqrt{25-9}} = -\frac{-6}{2\sqrt{16}} = \frac{6}{8} = \frac{3}{4}$$

We can solve this problem more easily by differentiable the given equation of the circle implicitly with respect to x :

$$\frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(y^2\right) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

The slope at (3,-4) is $-\left.\frac{x}{y}\right|_{(3,-4)} = -\frac{3}{-4} = \frac{3}{4}$

Example 23 : find $\frac{dy}{dx}$ if $y^2 = x^2 + \sin(xy)$.

Solution

We differentiate the equation implicitly

$$y^2 = x^2 + \sin(xy)$$

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}(\sin(xy))$$

$$2y\frac{d}{dx} = 2x + (\cos(xy)) \cdot \frac{d}{dx}(xy)$$

$$2y\frac{d}{dx} = 2x + (\cos(xy)) \cdot \left(y + x\frac{dy}{dx}\right)$$

$$2y\frac{dy}{dx}-(\cos(xy))\cdot\left(x\frac{dy}{dx}\right)=2x+y\cos(xy)$$

$$[2y-x(\cos(xy))]\cdot\left(\frac{dy}{dx}\right)=2x+y\cos(xy)$$

$$\frac{dy}{dx}=\frac{2x+y\cos(xy)}{2y-x\cos(xy)}$$

Derivatives of Higher Order

Implicit differentiation can also be used to find higher derivatives.

Example 24 : find $\frac{d^2y}{dx^2}$ if $2x^3-3y^2=8$.

Solution

To start, we differentiable both sides of the equation with respect to x in order to find $y'=\frac{dy}{dx}$.

$$\frac{d}{dx}\left(2x^3-3y^2\right)=\frac{d}{dx}(8)$$

$$6x^2-6yy'=0$$

$$y'=\frac{x^2}{y} \qquad when \qquad y\neq 0$$

We now apply the quotient rule to find y''

$$y''=\frac{d}{dx}\left(\frac{x^2}{y}\right)=\frac{2xy-x^2y'}{y^2}=\frac{2x}{y}-\frac{x^2}{y^2}y'$$

We substitute $y'=\frac{x^2}{y}$ to express y'' in terms of x and y.

$$y''=\frac{2x}{y}-\frac{x^2}{y^2}\cdot\frac{x^2}{y}=\frac{2x}{y}-\frac{x^4}{y^3} \qquad when \qquad y\neq 0$$

<u>Example 25 :</u> show that the point (2, 4) lies on the curve $x^3 + y^3 - 9xy = 0$. Then find the tangent.

<u>Solution</u>

The point (2, 4) lies on the curve because its coordinates satisfy the equation given for the curve : $2^3 + 4^3 - 9\cdot(2)\cdot(4) = 8 + 64 - 72 = 0$.

To find the slope of the curve at (2, 4), we first use implicit differentiation to find a formula for $\frac{dy}{dx}$:

$$x^3 + y^3 - 9xy = 0$$

$$\frac{d}{dx}(x^3) + \frac{d}{dx}(y^3) - \frac{d}{dx}(9xy) = \frac{d}{dx}(0)$$

$$3x^2 + 3y^2\frac{dy}{dx} - 9\left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) = 0$$

$$(3y^2 - 9x)\frac{dy}{dx} + 3x^2 - 9y = 0$$

$$3(y^2 - 3x)\frac{dy}{dx} = 3(3y - x^2)$$

$$\frac{dy}{dx} = \frac{(3y - x^2)}{(y^2 - 3x)}$$

We then evaluate the derivative at (x, y)= (2, 4):

$$\left.\frac{dy}{dx}\right|_{(2,4)} = \left.\frac{(3y - x^2)}{(y^2 - 3x)}\right|_{(2,4)} = \frac{(3\cdot(4) - 2^2)}{(4^2 - 3\cdot(2))} = \frac{(12-4)}{(16-6)} = \frac{8}{10} = \frac{4}{5}$$

The tangent at (2, 4) is the line through (2, 4) with slope (4/5) :

$$\frac{y - y_0}{x - x_0} = \frac{4}{5}$$

$$\frac{(y-4)}{(x-2)} = \frac{4}{5} \qquad \Rightarrow (y-4) = \left(\frac{4}{5}\right)\cdot(x-2)$$

$$y = 4 + \left(\frac{4}{5}\right)\cdot(x-2) \qquad \Rightarrow y = \frac{4}{5}x + \frac{12}{5}$$