<u>4.1 Sigma Notation and limits of Finite Sums</u>

Sigma notation enables us to write a sum with many terms in the compact form :

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + ... + a_{n-1} + a_n$$

The Greek letter $\sum$ stands for "sum". The index of summation k tells us where the sum begins. Any letter can be used to denote the index, but the letters i, j, and k are customary.

The index k ends at $k = n$

The summation symbol $\sum_{k=1}^{n} a_k$ is a formula for the kth term.

The index k starts at k=1

Thus we can write

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 = \sum_{k=1}^{11} k^2 \quad \text{and}$$

$$f(1) + f(2) + f(3) + ... + f(100) = \sum_{i=1}^{100} f(i)$$

The lower limit of summation does not have to be 1; it can be any integer.

<u>Example 1 :</u>

A sum in sigma notation	**The sum written out**	**The value of the sum**
$\sum_{k=1}^{5} k$	$1+2+3+4+5$	15
$\sum_{k=1}^{3} (-1)^k k$	$(-1)^1 \cdot 1 + (-1)^2 \cdot 2 + (-1)^3 \cdot 3$	-2
$\sum_{k=4}^{5} \left(\frac{k^2}{k-1} \right)$	$\left(\frac{4^2}{4-1} + \frac{5^2}{5-1} \right)$	$\left(\frac{16}{3} + \frac{25}{4} \right) = \frac{139}{12}$

Example 2 :

Express the sum $(1+3+5+7+9)$ in sigma notation.

Solution

The formula generating the terms changes with the lower limit of summation, it is often simplest to start with $k=0$ or $k=1$, but we can start with any integer.

starting with $k=0$; $1+3+5+7+9=\sum_{k=0}^{4}(2k+1)$

starting with $k=1$; $1+3+5+7+9=\sum_{k=1}^{5}(2k-1)$

starting with $k=2$; $1+3+5+7+9=\sum_{k=2}^{6}(2k-3)$

starting with $k=-3$; $1+3+5+7+9=\sum_{k=-3}^{1}(2k+7)$

Algebra Rules for finite Sums

1. Sum Rule: $\sum_{k=1}^{n}(a_k+b_k)=\sum_{k=1}^{n}(a_k)+\sum_{k=1}^{n}(b_k)$
2. Difference Rule: $\sum_{k=1}^{n}(a_k-b_k)=\sum_{k=1}^{n}(a_k)-\sum_{k=1}^{n}(b_k)$
3. Constant Multiple Rule: $\sum_{k=1}^{n}ca_k=c\cdot\sum_{k=1}^{n}(a_k)$
4. Constant Value Rule: $\sum_{k=1}^{n}c=n\cdot c$

Example 3 :

We demonstrate the use of the algebra rules.

(a) $\sum_{k=1}^{n}(3k-k^2)=3\cdot\sum_{k=1}^{n}k-\sum_{k=1}^{n}k^2$

(b) $\sum_{k=1}^{n}(-a_k)=\sum_{k=1}^{n}(-1)\cdot a_k=-1\cdot\sum_{k=1}^{n}(a_k)$

(c)
$$\sum_{k=1}^{3}(k+4)=\sum_{k=1}^{3}k+\sum_{k=1}^{3}4$$
$$=(1+2+3)+(3\cdot 4)$$
$$=6+12=18$$

(d) $\sum_{k=1}^{n}\left(\frac{1}{n}\right)=n\cdot\left(\frac{1}{n}\right)=1$

Example 4 :

Show that the sum of the first n integer is $\sum_{k=1}^{n}k=\left(\frac{n(n+1)}{2}\right)$

Solution

The formula tells us that the sum of the first 4 integers is

$$\left(\frac{n(n+1)}{2}\right)=\frac{(4)\cdot(5)}{2}=10$$

Addition verifies this prediction:

$$\sum_{k=1}^{n}k=(1+2+3+4)=10$$

The formulas for the sums of the square and cubes of the first n integers are proved using mathematical induction.

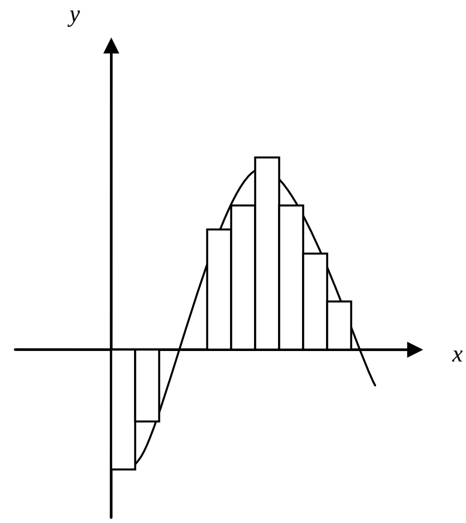
The first n squares : $\sum_{k=1}^{n}k^2=\left(\frac{n(n+1)(2n+1)}{6}\right)$

The first n cubes : $\sum_{k=1}^{n}k^3=\left(\frac{n(n+1)}{2}\right)^2$

4.2 The definite Integral

Let $f(x)$ be a function defined on a closed interval $[a,b]$. We say that a number J is the **definite integral of** f **over** $[a,b]$ and that J is the limit of the **Riemann sums** $\left(\sum_{k=1}^{n}f(c_k)\cdot\Delta x_k\right)$(the Area under any curve, see fig.). In the cases where the subintervals all have equal width $\Delta x = (b-a)/n$, we can form each **Riemann sum** as

$$S_n = \left(\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k \right) = \left(\sum_{k=1}^{n} f(c_k) \cdot \left(\frac{b-a}{n} \right) \right) \qquad \Delta x_k = \Delta x = (b-a)/n \text{ for all } k$$



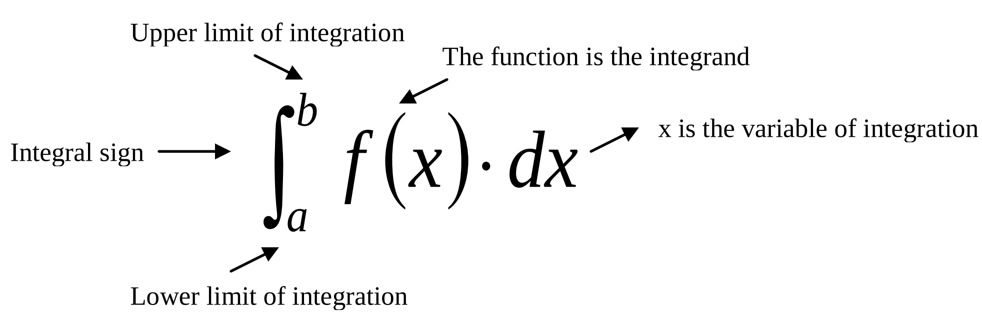
$$J = \lim_{n \to \infty} \left(\sum_{k=1}^{n} f(c_k) \cdot \left(\frac{b-a}{n} \right) \right) = \lim_{n \to \infty} \left(\sum_{k=1}^{n} f(c_k) \cdot \Delta x \right) \qquad \Delta x = (b-a)/n$$

$\left(\sum_{k=1}^{n} f(c_k) \cdot \Delta x_k \right)$ becoming an infinite sum of function values $f(x)$ multiplied by "infinitesimal" subinterval width dx. The sum symbol $\sum$ is replaced in the limit by the integer symbol $\int$, as x goes from a to b.

The symbol for the number J in the definition of the definite integral is

$$J = \int_a^b f(x) \cdot dx$$

The component parts in the integral symbol also have names:



Properties of definite integrals

Theorem 1 When f and g are integral function over the interval $[a,b]$, the definite integral satisfies the rules as follows:

1. Order of Integration: $\int_b^a f(x)\cdot dx = -\int_a^b f(x)\cdot dx$
2. Zero width interval: $\int_a^a f(x)\cdot dx = 0$
3. Constant multiple: $\int_a^b kf(x)\cdot dx = k\int_a^b f(x)\cdot dx$
4. Sum and Difference: $\int_a^b (f(x)\pm g(x))\cdot dx = \int_a^b f(x)\cdot dx \pm \int_a^b g(x)\cdot dx$
5. Additively: $\int_a^b f(x)\cdot dx + \int_b^c f(x)\cdot dx = \int_a^c f(x)\cdot dx$
6. Max-Min inequality: if f has maximum value max. f and minimum value min. f on $[a,b]$, then

$$\min.f\cdot(b-a) \le \int_a^b f(x)\cdot dx \le \max.f\cdot(b-a)$$

7. Domination: $f(x)\ge g(x) \quad on[a,b] \Rightarrow \int_a^b f(x)\cdot dx \ge \int_a^b g(x)\cdot dx$

$$f(x)\ge 0 \quad on[a,b] \Rightarrow \int_a^b f(x)\cdot dx \ge 0$$

Definition if $y=f(x)$ is nonnegative and integrable over a closed interval $[a,b]$, then the **area under the curve** $y=f(x)$ **over** $[a,b]$ is the integral of f from a to b.

$$A = \int_a^b f(x)\cdot dx$$

Leading us to define the **average value** as the area under the graph of $y=f(x)$divided by $(b-a)$, and written as:

$$Average = \frac{1}{(b-a)}\int_a^b f(x)\cdot dx$$

And also called **Mean value** or $av.(f)$.

Example 5 :

Find the average value of $f(x)=\sqrt{4-x^2}$ on $[-2,2]$.

Solution

The function above represent a graph is the upper semi-circle of radius 2 centered at the origin. The Area of semi-circle can be calculated with x-axis between -2 and 2 given as:

$$Area=\frac{1}{2}\cdot\pi r^2=\frac{1}{2}\cdot\pi(2)^2=2\pi$$

The Area is also the value of the integral of f from -2 to 2,

$$Area=\int_{-2}^{2}\sqrt{4-x^2}\cdot dx=2\pi$$

Therefore, the average value of f is

$$av.(f)=\frac{1}{2-(-2)}\int_{-2}^{2}\sqrt{4-x^2}\cdot dx=\frac{1}{4}\cdot(2\pi)=\frac{\pi}{2}$$

Theorem 2 if f is continuous on $[a,b]$, then at some point c in $[a,b]$,

$$f(c)=\frac{1}{(b-a)}\int_{a}^{b}f(x)\cdot dx$$

Theorem 3 if f is continuous on $[a,b]$, then $F(x)=\int_{a}^{x}f(t)\cdot dt$ is continuous on $[a,b]$ and differentiable on (a,b) and its derivative is $f(x)$:

$$F'(x)=\frac{d}{dx}\int_{a}^{x}f(t)\cdot dt=f(x)$$

Example 6 :

Use the fundamental theorem to find dy/dx if

(a) $y=\int_{a}^{x}(t^3+1)\cdot dt$ (b) $y=\int_{x}^{5}3t\cdot(\sin t)\cdot dt$ (c) $y=\int_{1}^{x^2}\cos t\cdot dt$

Solution

We calculate the derivatives with respect to the independent variable x.

(a) $\frac{dy}{dx}=\frac{d}{dx}\int_a^x\left(t^3+1\right)\cdot dt=x^3+1$

(b)

$$\frac{dy}{dx}=\frac{d}{dx}\int_x^5 3t\cdot(\sin t)\cdot dt=\frac{d}{dx}\left(-\int_5^x 3t\cdot(\sin t)\cdot dt\right)$$
$$=-\frac{d}{dx}\int_5^x 3t\cdot(\sin t)\cdot dt$$
$$=-3x\cdot(\sin x)$$

(c) the upper limit of integration is x^2, this makes y a composite of the two functions, $y=\int_1^u \cos t\cdot dt$ and $u=x^2$.

Therefore apply the chain rule to find $\frac{dy}{dx}\Rightarrow\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$.

$$\frac{dy}{dx}=\left(\frac{d}{du}\int_1^u \cos t\cdot dt\right)\cdot\frac{du}{dx}$$
$$=\cos u\cdot\frac{du}{dx}$$
$$=\cos x^2\cdot 2x$$
$$=2x\cos x^2$$

Theorem 4 if f is continuous at every point in $[a,b]$ and $F(x)$ is any anti-derivative of f on $[a,b]$, then

$$\int_a^b f(x)\cdot dx=F(b)-F(a)$$

Example 7 :

We calculate several definite integrals using the Evaluation theorem, rather than by taking limits of Riemann sums.

(a) $\int_0^\pi \cos x dx=\sin x\Big]_0^\pi$

$$=\sin\pi-\sin 0=0-0=0$$

(b) $\int_{-\pi/4}^{0} \sec x \tan x dx = \sec x \Big]_{-\pi/4}^{0}$

$$= \sec 0 - \sec(-\pi/4) = 1 - \sqrt{2}$$

(c)

$$\int_1^4 \left(\frac{3}{2}\sqrt{x} - \frac{4}{x^2}\right)dx = \left[x^{3/2} + \frac{4}{x}\right]_1^4$$

$$= \left[4^{3/2} + \frac{4}{4}\right] - \left[1^{3/2} + \frac{4}{1}\right] = [8 + 1 - 1 - 4] = 4$$

<u>Theorem 5</u> the net change in a function $F(x)$ over an interval $a \le x \le b$ is the integral of its rate of change: $F(b) - F(a) = \int_a^b F'(x) \cdot dx$

<u>Example 8 :</u> the velocity of the rock at any time *t* during its motion was given as $v(t) = 49 - 9.8(t)$ *(m/s)*

a) find the displacement of the rock during the time period $0 \le t \le 8$.

b) find the total distance traveled during this time period.

<u>Solution</u>

a) the displacement is the integral

$$\int_0^8 v(t) \cdot dt = \int_0^8 (49 - 9.8t) \cdot dt = \left[49t - 4.9t^2\right]_0^8$$

$$= 49 \cdot (8) - 4.9 \cdot (8)^2 = 78.4(m/s)$$

b) the velocity function $v(t)$ is positive over the time interval *[0,5]* and negative over the interval *[5,8]*. Therefore , the total distance traveled is the integral

$$\int_0^8 |v(t)| \cdot dt = \int_0^5 |v(t)| \cdot dt + \int_5^8 |v(t)| \cdot dt$$

$$= \int_0^5 (49 - 9.8t) \cdot dt - \int_5^8 (49 - 9.8t) \cdot dt$$

$$= \left[49t - 4.9t^2\right]_0^5 - \left[49t - 4.9t^2\right]_5^8$$

$$= [49(5) - 4.9 \cdot (25)] - [49(8) - 4.9 \cdot (64) - (49(5) - 4.9 \cdot (25))]$$

$$= 122.5 - (-44.1) = 166.6m$$

Example 9 : which function $f(x)=\sin(x)$ between $x=0$ and $x=2\pi$.

Compute :

a) the definite integral of $f(x)$ over $[0,2\pi]$.

b) The area between the graph of $f(x)$ and the x-axis over $[0,2\pi]$.

Solution

the definite integral for $f(x)=\sin x$ is given by

the definite integral of $\int_0^{2\pi} \sin x \cdot dx = -[\cos x]_0^{2\pi} = -[\cos 2\pi - \cos 0] = -[1-1] = 0$

the area between the graph of $f(x)$ and the x-axis over $[0,2\pi]$ is calculated by breaking up the domain of $\sin(x)$ into two pieces: the interval $[0,\pi]$ over which it is non-negative and the interval $[\pi,2\pi]$ over which it is non-positive.

$$\int_0^{\pi} \sin x \cdot dx = -[\cos x]_0^{\pi} = -[\cos \pi - \cos 0] = -[-1-1] = 2$$

$$\int_{\pi}^{2\pi} \sin x \cdot dx = -[\cos x]_{\pi}^{2\pi} = -[\cos 2\pi - \cos \pi] = -[1-(-1)] = -2$$

The area between the graph of and the axis is obtained by adding the absolute values

$A = |2| + |-2| = 4$

Summary:

To find the area between the graph of $f(x)$ and the x-axis over the interval *[a, b]* :

1. Subdivide *[a, b]* at the zeros of *f*.
2. Integrate *f* over each subinterval.
3. Add the Absolute values of the integrals.

<u>Example 10 :</u> find the Area of the region between the x-axis and the graph of $f(x)=x^3-x^2-2x\ ,\ -1\le x\le 2$

<u>Solution</u>

First find the zero of f. since

$f(x)=x^3-x^2-2x=x(x^2-x-2)=x(x+1)(x-2)$

The zero are *x=0,-1, and 2*. the zeros subdivide *[-1, 2]* into two subintervals: *[-1, 0]* ,on which $f\ge 0$, and *[0, 2]*, on which $f\le 0$, we integrate f over each subinterval and add the absolute values of the calculated integrals.

$$\int_{-1}^{0}(x^3-x^2-2x)\cdot dx=\left[\frac{x^4}{4}-\frac{x^3}{3}-x^2\right]_{-1}^{0}=0-\left[\frac{1}{4}+\frac{1}{3}-1\right]=\frac{5}{12}$$

$$\int_{0}^{2}(x^3-x^2-2x)\cdot dx=\left[\frac{x^4}{4}-\frac{x^3}{3}-x^2\right]_{0}^{2}=\left[4-\frac{8}{3}-4\right]-0=-\frac{8}{3}$$

Total enclosed area $=\left|\frac{5}{12}\right|+\left|-\frac{8}{3}\right|=\frac{37}{12}$

<u>4.3 Indefinite Integrals and the substitution method</u>

We defined the **indefinite integral** of the function f with respect to x as the set of all anti-derivatives of f, symbolized by

$$\int f(x)\cdot dx$$

Since any two anti-derivatives of f differ by a constant, the indefinite integral $\int$ notation means that for any anti-derivative F of f,

$$\int f(x)\cdot dx=F(x)+C$$

Where C is any arbitrary constant.

Basic of Indefinite integration formulas

1. $\int k \cdot dx = k \cdot x + C$ (any number k)

2. $\int x^n \cdot dx = \dfrac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$

3. $\int \dfrac{dx}{x} = \ln|x| + C$

4. $\int e^x \cdot dx = e^x + C$

5. $\int a^x \cdot dx = \dfrac{a^x}{\ln a} + C \qquad (a > 0, a \neq 1)$

6. $\int \sin x \cdot dx = -\cos x + C$

7. $\int \cos x \cdot dx = \sin x + C$

8. $\int \sec^2 x \cdot dx = \tan x + C$

9. $\int \csc^2 x \cdot dx = -\cot x + C$

10. $\int \sec x \cdot \tan x \cdot dx = \sec x + C$

11. $\int \csc x \cdot \cot x \cdot dx = -\csc x + C$

12. $\int \tan x \cdot dx = \ln|\sec x| + C$

13. $\int \cot x \cdot dx = \ln|\sin x| + C$

14. $\int \sec x \cdot dx = \ln|\sec x + \tan x| + C$

15. $\int \csc x \cdot dx = -\ln|\csc x + \cot x| + C$

16. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right) + C$

17. $\int \dfrac{dx}{(a^2 + x^2)} = \dfrac{1}{a} \cdot \tan^{-1}\left(\dfrac{x}{a}\right) + C$

18. $\int \dfrac{dx}{x \cdot \sqrt{(x^2 - a^2)}} = \dfrac{1}{a} \cdot \sec^{-1}\left|\dfrac{x}{a}\right| + C$

Substitution: Running the chain rule

If u is a differentiable function of x and n is any number different from -1, the chain rule tells us that

$$\frac{d}{dx}\left(\frac{u^{n+1}}{n+1}\right)=u^n\frac{du}{dx}$$

Therefore,

$$\int u^n\frac{du}{dx}\cdot dx=\left(\frac{u^{n+1}}{n+1}\right)+C$$

And go to the simpler integral

$$\int u^n\cdot du=\left(\frac{u^{n+1}}{n+1}\right)+C$$

Example 1 : Find the integral $\int (x^3+x)^5\cdot(3x^2+1)dx$

Solution We set $u=x^3+x$ then

$du=\frac{du}{dx}\cdot dx=(3x^2+1)\cdot dx$, so that by substitution we have

$$\int (x^3+x)^5\cdot\underline{(3x^2+1)\cdot dx}=\int u^5du$$

$$=\frac{u^6}{6}+C$$

$$=\frac{(x^3+x)^6}{6}+C$$

Example 2 : Find the integral $\int\sqrt{(2x+1)}\cdot dx$

Solution the integral dose not fit the formula $\int u^n du$, with $u=2x+1$ and $n=1/2$, because

$$du=\frac{du}{dx}\cdot dx=2\cdot dx$$

So we write

$$\int \sqrt{(2x+1)}\cdot dx = \frac{1}{2}\cdot\int\sqrt{(2x+1)}\cdot 2dx$$
$$= \frac{1}{2}\cdot\int u^{1/2}du$$
$$= \frac{1}{2}\cdot\frac{u^{3/2}}{3/2}+C$$
$$= \frac{1}{3}\cdot(2x+1)^{3/2}+C$$

<u>**Theorem 6**</u> **– The substitution Rule** if $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x))\cdot g'(x)\cdot dx = \int f(u)\cdot du$$

Proof :

By the chain rule, $F(g(x))$ is an antiderivative of $f(g(x))\cdot g'(x)$ whenever F is an antiderivative of f :

$$\frac{d}{dx}F(g(x)) = F'(g(x))\cdot g'(x) \qquad F' = f$$
$$= f(g(x))\cdot g'(x)$$

If we make the substitution $u = g(x)$, then

$$\int f(g(x))\cdot g'(x)\cdot dx = \int \frac{d}{dx}F(g(x))\cdot dx$$
$$= F(g(x))+C$$
$$= F(u)+C$$
$$= \int F'(u)du$$
$$= \int f(u)du$$

The Substitution Rule provides the following **Substitution method** to evaluate the integral $\int f(g(x))\cdot g'(x)dx$, When f and g' are continuous functions :

1. Substitute $u = g(x)$ and $du = \left(\frac{du}{dx}\right)\cdot dx = g'(x)dx$ to obtain the integral

   $\int f(u)\cdot du$

2. integrate with respect to u.
3. Replace u by $g(x)$ in the result.

Example 3 : Find $\int \sec^2(5t+1)\cdot 5\cdot dt$

Solution we substitute $u=(5t+1)$ and $du = 5\cdot dt$. then,

$$\begin{aligned}\int \sec^2(5t+1)\cdot 5\cdot dt &= \int \sec^2(u)\cdot du\\ &= \tan u + C\\ &= \tan(5t+1)+C\end{aligned}$$

Example 4 : Find $\int \cos(7\theta+3)\cdot d\theta$

Solution we let $u=(7\theta+3)$ so that $du = 7\cdot d\theta$, therefore, multiplying and dividing by 7,

$$\begin{aligned}\int \cos(7\theta+3)\cdot d\theta &= \frac{1}{7}\int \cos(7\theta+3)\cdot 7d\theta\\ &= \frac{1}{7}\int \cos(u)\cdot du\\ &= \frac{1}{7}\cdot \sin u + C\\ &= \frac{1}{7}\cdot \sin(7\theta+3)+C\end{aligned}$$

Example 5 : Find the integral $\int x^2\cdot \sin(x^3)\cdot dx$

Solution we let $u=(x^3)$ so that $du = 3x^2\cdot dx \Rightarrow x^2dx = (1/3)\cdot du$, therefore

$$\int x^2 \cdot \sin(x^3) \cdot dx = \int \sin u \cdot (1/3) \cdot du$$
$$= \frac{1}{3}\int \sin u \cdot du$$
$$= \frac{1}{3} \cdot (-\cos u) + C$$
$$= -\frac{1}{3} \cdot \cos(x^3) + C$$

Example 6 : Evaluate $\int x \cdot \sqrt{2x+1} \cdot dx$

Solution we let $u = 2x+1$ so that $du = 2 \cdot dx$. Then,

$\sqrt{2x+1} \cdot dx = \frac{1}{2} \cdot \sqrt{u} \cdot du$ and $u = 2x+1 \Rightarrow x = (u-1)/2$, therefore;

$x \cdot \sqrt{2x+1} \cdot dx = \frac{1}{2}(u-1) \cdot \frac{1}{2} \cdot \sqrt{u} \cdot du$ the integration now becomes

$$\int x \cdot \sqrt{2x+1} \cdot dx = \frac{1}{4}\int (u-1) \cdot \sqrt{u} \cdot du = \frac{1}{4}\int (u-1) \cdot u^{1/2} \cdot du$$
$$= \frac{1}{4}\int \left(u^{3/2} - u^{1/2}\right) \cdot du$$
$$= \frac{1}{4}\left(\frac{2u^{5/2}}{5} - \frac{2u^{3/2}}{3}\right) + C$$
$$= \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C$$

Example 7 : Evaluate $\int \frac{2z \cdot dz}{\sqrt[3]{z^2+1}}$

we might try $u = z^2 + 1$ or we take $u = \sqrt[3]{z^2+1}$.

Solution 1 substitute $u = z^2 + 1$ and $du = 2z \cdot dz$

$$\int \frac{2z \cdot dz}{\sqrt[3]{z^2+1}} = \int \frac{du}{u^{1/3}}$$
$$= \int u^{-1/3} \cdot du$$
$$= \frac{u^{2/3}}{2/3} + C$$
$$= \frac{3}{2}\left(z^2+1\right)^{2/3} + C$$

<u>Solution 2</u> substitute $u=\sqrt[3]{z^2+1}$ and $u^3=z^2+1 \Rightarrow 3u^2\cdot du = 2z\cdot dz$

$$\int \frac{2z\cdot dz}{\sqrt[3]{z^2+1}} = \int \frac{3u^2\cdot du}{u}$$
$$= 3\int u\cdot du$$
$$= 3\cdot\frac{u^2}{2}+C$$
$$= \frac{3}{2}\left(z^2+1\right)^{2/3}+C$$

The integrals of $\sin^2 x$ **and** $\cos^2 x$

Sometimes we can use trigonometric identities to transform integrals we do not know how to evaluate into ones we can evaluate using the substitution rule.

<u>Example 8 :</u>

a)

$$\int \sin^2 x\cdot dx = \int \frac{1-\cos 2x}{2}\cdot dx$$
$$= \frac{1}{2}\int (1-\cos 2x)\cdot dx$$
$$= \frac{1}{2}\left(x-\frac{\sin 2x}{2}\right)+C$$
$$= \frac{x}{2}-\frac{\sin 2x}{4}+C$$

b)

$$\int \cos^2 x\cdot dx = \int \frac{1+\cos 2x}{2}\cdot dx$$
$$= \frac{1}{2}\int (1+\cos 2x)\cdot dx$$
$$= \frac{1}{2}\left(x+\frac{\sin 2x}{2}\right)+C$$
$$= \frac{x}{2}+\frac{\sin 2x}{4}+C$$

Example 9 :

$$\int \sqrt{x} \cdot \sin^2\left(x^{3/2} - 1\right) \cdot dx \quad , \quad u = \left(x^{3/2} - 1\right)$$

Solution

$$du = \frac{3}{2} \cdot x^{1/2} \cdot dx$$

$$\int \sqrt{x} \cdot \sin^2\left(x^{3/2} - 1\right) \cdot dx = \frac{2}{3} \int \sin^2\left(x^{3/2} - 1\right) \cdot \left(\frac{3}{2}\sqrt{x} \cdot dx\right)$$

$$= \frac{2}{3} \int \sin^2(u) \cdot du$$

When $\sin^2 x = \frac{1 - \cos(2x)}{2}$, therefore

$$\frac{2}{3} \int \sin^2(u) \cdot du = \frac{2}{3} \int \frac{1 - \cos(2u)}{2}$$

$$= \frac{1}{3} \int (1 - \cos 2u) \cdot du$$

$$= \frac{1}{3}\left(u - \frac{\sin 2u}{2}\right) + C$$

$$= \frac{u}{3} - \frac{\sin 2u}{6} + C$$

$$= \frac{\left(x^{3/2} - 1\right)}{3} - \frac{\sin\left(2x^{3/2} - 2\right)}{6} + C$$

Theorem 7 – **Substitution in definite integrals** if g' is continuous on the interval $[a,b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x)) \cdot g'(x) \cdot dx = \int_{g(a)}^{g(b)} f(u) \cdot du$$

When $g(b) = u, when \Rightarrow x = b$

And $g(a) = u, when \Rightarrow x = a$

Example 10 : Evaluate $\int_{-1}^{1} 3x^2\left(\sqrt{x^3+1}\right)\cdot dx$

Solution

Let $u = x^3 + 1$, $du = 3x^2 dx$

When $x = -1$, $u = (-1)^3 + 1 = 0$

When $x = 1$, $u = (1)^3 + 1 = 2$, then

$$\int_{-1}^{1} 3x^2\left(\sqrt{x^3+1}\right)\cdot dx = \int_{0}^{2} \sqrt{u}\cdot du$$

$$= \int_{0}^{2} u^{1/2}\cdot du$$

$$= \left[\frac{2}{3}u^{3/2}\right]_0^2$$

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

Example 11 : Evaluate $\int_{\pi/4}^{\pi/2} \cot\theta\cdot\csc^2\theta\cdot d\theta$

Solution

Let $u = \cot\theta$, $du = -\csc^2\theta\cdot d\theta \Rightarrow -du = \csc^2\theta\cdot d\theta$

When $\theta = \pi/4$, $u = \cot(\pi/4) = 1$

When $\theta = \pi/2$, $u = \cot(\pi/2) = 0$, then

$$\int_{\pi/4}^{\pi/2} \cot\theta\cdot\csc^2\theta\cdot d\theta = \int_{1}^{0} u\cdot(-du)$$

$$= -\int_{1}^{0} u\cdot du$$

$$= -\left[\frac{u^2}{2}\right]_1^0$$

$$= -\left[\frac{(0)^2}{2} - \frac{(1)^2}{2}\right] = \frac{1}{2}$$

Theorem 8 – **Definite integrals of Symmetric functions** let f be continuous on the Symmetric interval $-[a,a]$.

a) if f is even, then $\int_{-a}^{a} f(x)\cdot dx = 2\int_{0}^{a} f(x)\cdot dx$

b) if f is odd, then $\int_{-a}^{a} f(x)\cdot dx = 0$

Proof

$$\int_{-a}^{a} f(x)\cdot dx = \int_{-a}^{0} f(x)\cdot dx + \int_{0}^{a} f(x)\cdot dx$$

$$= -\int_{0}^{-a} f(x)\cdot dx + \int_{0}^{a} f(x)\cdot dx$$

Let $u = -x, du = -dx$

When $x = 0 \Rightarrow u = 0$ and $x = -a \Rightarrow u = a$, then

$$\int_{-a}^{a} f(x)\cdot dx = -\int_{0}^{a} f(-u)\cdot(-du) + \int_{0}^{a} f(x)\cdot dx$$

$$= \int_{0}^{a} f(-u)\cdot du + \int_{0}^{a} f(x)\cdot dx$$

a) If f is even, so $f(-u) = f(u)$, then

$$\int_{-a}^{a} f(x)\cdot dx = \int_{0}^{a} f(-u)\cdot du + \int_{0}^{a} f(x)\cdot dx$$

$$= \int_{0}^{a} f(u)\cdot du + \int_{0}^{a} f(x)\cdot dx$$

$$= 2\int_{0}^{a} f(x)\cdot dx$$

b) If f is odd, so $f(-u) = -f(u)$, then

$$\int_{-a}^{a} f(x)\cdot dx = \int_{0}^{a} f(-u)\cdot du + \int_{0}^{a} f(x)\cdot dx$$

$$= -\int_{0}^{a} f(u)\cdot du + \int_{0}^{a} f(x)\cdot dx = 0$$

<u>Example 12 :</u> Evaluate $\int_{-2}^{2}\left(x^4-4x^2+6\right)\cdot dx$

<u>Solution</u>

Since $f(x)=x^4-4x^2+6$ satisfies $f(-x)=f(x)$, it is even on the symmetric interval *[-2,2]* , so

$$\int_{-2}^{2}\left(x^4-4x^2+6\right)\cdot dx=2\int_{0}^{2}\left(x^4-4x^2+6\right)\cdot dx$$

$$=2\left[\frac{x^5}{5}-\frac{4}{3}x^3+6x\right]_0^2$$

$$=2\left(\frac{32}{5}-\frac{32}{3}+12\right)=2\left(\frac{32*3-32*5+12*15}{15}\right)=\left(\frac{232}{15}\right)$$

<u>4.4 Techniques of Integration</u>

We study a number of other important techniques for finding antiderivatives for many combinations of functions whose antiderivatives cannot be found using the methods presented before.

a. Integration by parts

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)\cdot g(x)\cdot dx$$

It is useful when f can be differentiated repeatedly and g can be integrated repeatedly without difficulty. The integrals

$\int x\cdot\cos(x)\cdot dx$, $\int x^2\cdot e^x\cdot dx$ and $\int e^x\cdot\cos(x)\cdot dx$

Let $u=f(x)$ and $v=g(x)$. Then $du=f'(x)dx$ and $dv=g'(x)dx$. Using the Substitution Rule, the integration by parts formula becomes

$$\int u \cdot dv = u \cdot v - \int v \cdot du$$

To avoid mistakes, we always list our choices for u and dv, then we add to the list our calculated new terms du *and* v, and finally we apply the above formula

Example 13 : find $\int x \cdot \cos(x) \cdot dx$

Solution we use the formula $\int u \cdot dv = u \cdot v - \int v \cdot du$

$u = x \quad , \quad dv = \cos(x)dx$

$du = dx \quad , \quad v = \sin(x) \quad$, then

$$\int x \cdot \cos(x) \cdot dx = x \cdot \sin(x) - \int \sin(x)dx$$
$$= x \cdot \sin(x) + \cos(x) + C$$

These are four choices available for u and dv in ex.13 :

1. let $u = 1$ and $dv = x \cdot \cos(x)dx$ 2. let $u = x$ and $dv = \cos(x)dx$

3. let $u = x \cdot \cos(x)$ and $dv = dx$ 4. let $u = \cos(x)$ and $dv = x \cdot dx$

Example 14 : find $\int \ln(x) \cdot dx$

Solution since $\int \ln(x) \cdot dx$ can be written as $\int \ln(x) \cdot 1 \cdot dx$, we use the formula

$\int u \cdot dv = u \cdot v - \int v \cdot du$

$u = \ln(x) \quad , \quad dv = dx$

$du = \frac{1}{x}dx \quad , \quad v = x \quad$, then

$$\int \ln(x) \cdot dx = x \cdot \ln(x) - \int x \cdot \frac{1}{x} \cdot dx$$
$$= x \cdot \ln(x) - \int dx$$
$$= x \cdot \ln(x) - x + C$$

Example 15 : find $\int x^2 \cdot e^x \cdot dx$

Solution Let

$u = x^2 \quad , \quad dv = e^x dx$

$du = 2xdx \quad , \quad v = e^x$, then

$\int x^2 \cdot e^x \cdot dx = x^2 \cdot e^x - 2\int x \cdot e^x dx$

Also we integrate the second term $\left(\int x \cdot e^x dx\right)$ by part

$u = x \quad , \quad dv = e^x dx$

$du = dx \quad , \quad v = e^x$, then

$$\int x \cdot e^x dx = x \cdot e^x - \int e^x dx$$
$$= x \cdot e^x - e^x + C$$

Using this last evaluation, we then obtain

$$\int x^2 \cdot e^x \cdot dx = x^2 \cdot e^x - 2\int x \cdot e^x dx$$
$$= x^2 \cdot e^x - 2\left(x \cdot e^x - e^x + C\right)$$
$$= x^2 \cdot e^x - 2x \cdot e^x + 2e^x + C$$

Example 16 : evaluate $\int e^x \cdot \cos(x) \cdot dx$

Solution Let

$u = e^x \quad , \quad dv = \cos(x) dx$

$du = e^x dx \quad , \quad v = \sin(x)$, then

$\int e^x \cdot \cos(x) \cdot dx = e^x \sin(x) - \int e^x \sin x dx$

Also we integrate the second term $\left(\int e^x \sin x dx\right)$ by part

$u = e^x \quad , \quad dv = \sin(x) dx$

$du = e^x dx \quad , \quad v = -\cos(x)$, then

$$\int e^x \sin x dx = -e^x \cos(x) - \int e^x(-\cos(x))dx$$
$$= -e^x \cos(x) + \int e^x \cos(x) dx$$

Using this last evaluation, we then obtain

$$\int e^x \cdot \cos(x) \cdot dx = e^x \sin(x) - \int e^x \sin x dx$$
$$= e^x \sin(x) - \left[-e^x \cos(x) + \int e^x \cos(x) dx\right]$$
$$= e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x) dx$$

The unknown integral now appears on both sides of the equation. Adding the integral to both sides and adding the constant of integration give

$$\int e^x \cdot \cos(x) \cdot dx + \int e^x \cdot \cos(x) \cdot dx = e^x \sin(x) + e^x \cos(x) - \int e^x \cos(x) dx + \int e^x \cdot \cos(x) \cdot dx$$
$$2\int e^x \cdot \cos(x) \cdot dx = e^x \sin(x) + e^x \cos(x) + C_1$$

Dividing by (2) and renaming the constant of integration give

$$\int e^x \cdot \cos(x) \cdot dx = \frac{e^x \sin(x) + e^x \cos(x)}{2} + C$$

Example 17 : obtain a formula that expresses the integral $\int \cos^n(x) \cdot dx$

Solution we may think of $\cos^n(x)$ as $\cos^{n-1}(x) \cdot \cos(x)$. Then we let

$u = \cos^{n-1}(x)$, $dv = \cos(x)dx$

$du = (n-1) \cdot \cos^{n-2}(x) \cdot (-\sin(x))dx$, $v = \sin(x)$, then

Integration by part then gives

$$\int \cos^n(x) \cdot dx = \cos^{n-1}(x) \cdot \sin(x) + (n-1) \cdot \int \sin^2(x) \cdot \cos^{n-2}(x) dx$$
$$= \cos^{n-1}(x) \cdot \sin(x) + (n-1) \cdot \int (1 - \cos^2(x)) \cdot \cos^{n-2}(x) dx$$
$$= \cos^{n-1}(x) \cdot \sin(x) + (n-1) \cdot \int \cos^{n-2}(x) dx - (n-1) \cdot \int \cos^n(x) dx$$

If we add $(n-1) \cdot \int \cos^n(x) dx$ to both sides of this equation, we obtain

$$\int \cos^n(x) \cdot dx + (n-1) \cdot \int \cos^n(x) dx = \cos^{n-1}(x) \cdot \sin(x) + (n-1) \cdot \int \cos^{n-2}(x) dx$$
$$-(n-1)\int \cos^n(x) dx + (n-1) \cdot \int \cos^n(x) dx$$

$n\cdot\int\cos^n(x)\cdot dx=\cos^{n-1}(x)\cdot\sin(x)+(n-1)\cdot\int\cos^{n-2}(x)dx$, dividing by (n) to obtain

$$\int\cos^n(x)\cdot dx=\frac{\cos^{n-1}(x)\cdot\sin(x)}{n}+\frac{(n-1)}{n}\cdot\int\cos^{n-2}(x)dx$$

We apply the formula repeatedly until the remaining integral is easy to evaluate. For example if *n=3*, the result tells us that

$$\int\cos^3(x)\cdot dx=\frac{\cos^2(x)\cdot\sin(x)}{3}+\frac{2}{3}\cdot\int\cos(x)dx$$
$$=\frac{1}{3}\cos^2(x)\cdot\sin(x)+\frac{2}{3}\cdot\sin(x)+C$$

Integration by parts Formula for definite Integrals

$$\int_a^b f(x)\cdot g'(x)\cdot dx=f(x)\cdot g(x)\Big|_a^b-\int_a^b f'(x)\cdot g(x)\cdot dx$$

<u>Example 18 :</u> find the area of the region bounded by the curve $y=xe^{-x}$ and the x-axis from *x=0* to *x=4.*

<u>Solution</u> the area of the region is

$\int_0^4 x\cdot e^{-x}dx$, and let

$u=x \quad , \quad dv=e^{-x}dx$

$du=dx \quad , \quad v=-e^{-x}$, then

$$\int_0^4 x\cdot e^{-x}dx=-x\cdot e^{-x}\Big|_0^4-\int_0^4\left(-e^{-x}\right)dx$$
$$=\left[-4e^{-4}-0\right]+\int_0^4 e^{-x}dx$$
$$=-4e^{-4}-e^{-x}\Big|_0^4$$
$$=-4e^{-4}-\left(e^{-4}-e^{0}\right)=-4e^{-4}-e^{-4}+1=1-5e^{-4}\approx 0.91$$

Example 19 : Evaluate $\int x^2 \cdot e^x dx$

Solution with $f(x)=x^2$ and $g(x)=e^x$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	(+)	e^x
$2x$	(-)	e^x
2	(+)	e^x
0		e^x

$$\int x^2 \cdot e^x dx = x^2 \cdot e^x - 2x \cdot e^x + 2e^x + C$$

Example 20 : Evaluate $\int x^3 \cdot \sin(x) dx$

Solution with $f(x)=x^3$ and $g(x)=\sin(x)$, we list:

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^3	(+)	$\sin(x)$
$3x^2$	(-)	$-\cos(x)$
$6x$	(+)	$-\sin(x)$
6	(-)	$\cos(x)$
0		$\sin(x)$

$$\int x^3 \cdot \sin(x) dx = -x^3 \cdot \cos(x) + 3x^2 \cdot \sin(x) + 6x \cdot \cos(x) - 6\sin(x) + C$$

b. Trigonometric Substitutions

Trigonometric substitutions occur when we replace the **variable of integration** by a trigonometric function. The most common substitutions are $x = a\tan(x)$, $x = a\sin(x)$ and $x = a\sec(x)$. These substitutions are effective in transforming integrals involving $\sqrt{a^2+x^2}$, $\sqrt{a^2-x^2}$ and $\sqrt{x^2-a^2}$ into integrals. When

$$x = a\tan(\theta)\ ,\quad \theta = \tan^{-1}\left(\frac{x}{a}\right) \text{ with } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$a^2 + x^2 = a^2 + a^2\tan^2(\theta) = a^2(1+\tan(\theta)) = a^2\sec^2(\theta)$$

$$x = a\sin(\theta)\ ,\quad \theta = \sin^{-1}\left(\frac{x}{a}\right) \text{ with } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

$$a^2 - x^2 = a^2 - a^2\sin^2(\theta) = a^2(1-\sin(\theta)) = a^2\cos^2(\theta)$$

$$x = a\sec(\theta)\ ,\quad \theta = \sec^{-1}\left(\frac{x}{a}\right) \text{ with } \begin{matrix} 0\le\theta\le\frac{\pi}{2} & if\ \frac{x}{a}\ge 1 \\ \frac{\pi}{2}\le\theta\le 0 & if\ \frac{x}{a}\le -1 \end{matrix}$$

$$x^2 - a^2 = a^2\sec^2(\theta) - a^2 = a^2\left(\sec^2(\theta)-1\right) = a^2\tan^2(\theta)$$

Example 21 : Evaluate $\int \frac{dx}{\sqrt{4+x^2}}$

Solution we let

$$x = 2\tan(\theta) \text{ and } dx = 2\sec^2(\theta)d\theta\ ,\ -\frac{\pi}{2}\langle\theta\langle\frac{\pi}{2}$$

$$4 + x^2 = 4 + 4\tan^2(\theta) = 4\left(1+\tan^2(\theta)\right) = 4\sec^2\theta\ , \text{ then}$$

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\cdot d\theta}{\sqrt{4\sec^2\theta}} = \int \frac{2\sec^2\theta\cdot d\theta}{2|\sec\theta|} = \int \frac{\sec^2\theta\cdot d\theta}{|\sec\theta|}$$

$$= \int \sec\theta\cdot d\theta = \ln|\sec(\theta)+\tan(\theta)| + C$$

$$= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C$$

$$\begin{pmatrix} \sqrt{\sec^2\theta} = |\sec\theta| \\ \sec\theta\rangle 0 \quad for -\frac{\pi}{2}\langle\theta\langle\frac{\pi}{2} \end{pmatrix}$$

<u>Example 22 :</u> Evaluate $\int \frac{x^2 dx}{\sqrt{9-x^2}}$

<u>Solution</u> we let

$x = 3\sin(\theta)$ and $dx = 3\cos(\theta)d\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$9 - x^2 = 9 - 9\sin^2(\theta) = 9(1-\sin^2(\theta)) = 9\cos^2\theta$, then

$$\int \frac{x^2 dx}{\sqrt{9-x^2}} = \int \frac{9\sin^2\theta \cdot 3\cos\theta d\theta}{|3\cos\theta|}$$

$$= 9\int \sin^2\theta d\theta = 9\int \frac{(1-\cos(2\theta))}{2} d\theta$$

$$= \frac{9}{2}\left(\theta - \frac{\sin(2\theta)}{2}\right) + C = \frac{9}{2}(\theta - \sin(\theta)\cdot\cos(\theta)) + C$$

$$= \frac{9}{2}\left(\sin^{-1}\left(\frac{x}{3}\right) - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + C$$

$$= \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{x}{2}\cdot\sqrt{9-x^2} + C$$

<u>Example 23 :</u> Evaluate $\int \frac{dx}{\sqrt{25x^2-4}}$, $x > \frac{2}{5}$

<u>Solution</u> we first rewrite the radical

$$\sqrt{25x^2-4} = \sqrt{25\left(x^2 - \frac{4}{25}\right)} = 5\sqrt{x^2 - \left(\frac{2}{5}\right)^2}$$

To put the radicand in the form $x^2 - a^2$, we then substitute

$x = \frac{2}{5}\sec(\theta)$ and $dx = \frac{2}{5}\sec(\theta)\cdot\tan(\theta)d\theta$, $0 < \theta < \frac{\pi}{2}$

$x^2 - \left(\frac{2}{5}\right)^2 = \frac{4}{25}\sec^2(\theta) - \frac{4}{25} = \frac{4}{25}(\sec^2(\theta) - 1) = \frac{4}{25}\tan^2(\theta)$, then

$$\sqrt{x^2 - \left(\frac{2}{5}\right)^2} = \frac{2}{5}|\tan(\theta)| = \frac{2}{5}\tan(\theta)$$

With these substitutions, we have

$$\int \frac{dx}{\sqrt{25x^2-4}} = \int \left[\frac{dx}{5\sqrt{x^2-\left(\frac{2}{5}\right)^2}} \right] = \int \left[\frac{\frac{2}{5}\sec(\theta)\cdot\tan(\theta)d\theta}{5\cdot\left(\frac{2}{5}\tan(\theta)\right)} \right]$$

$$= \frac{1}{5}\int \sec(\theta)\cdot d\theta = \frac{1}{5}\ln\left|\sec(\theta)+\tan(\theta)\right| + C$$

$$= \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2-4}}{2}\right| + C$$

c. Integration of Rational functions by Partial Fractions

d.