LECTURE NOTE

ON

PROBABILITY AND SATISTICS 2

BY

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LECTURE 4#

Some examples for some discrete distributions

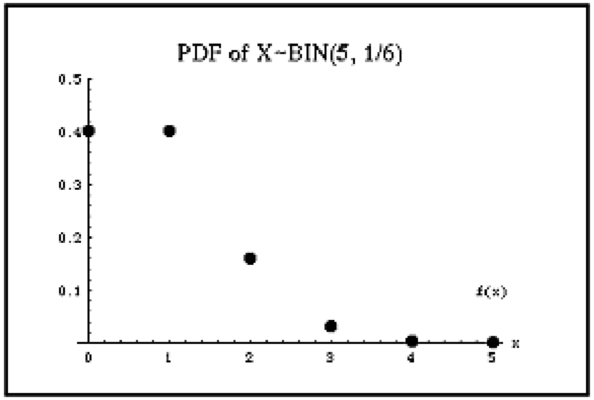
Example1

What is the probability of rolling at most two sixes in 5 independent casts of a fair die?

Sol:

Let the random variable X denote number of sixes in 5 independent casts of a fair die. Then X is a binomial random variable with probability of success p and n = 5. The probability of getting a six is p=$\frac{1}{6}$. Hence, the probability of rolling at most two sixes is:

$$
\begin{aligned}
P(X \leq 2) &= F(2) = f(0) + f(1) + f(2) \\
&= \binom{5}{0}\left(\frac{1}{6}\right)^{0}\left(\frac{5}{6}\right)^{5} + \binom{5}{1}\left(\frac{1}{6}\right)^{1}\left(\frac{5}{6}\right)^{4} + \binom{5}{2}\left(\frac{1}{6}\right)^{2}\left(\frac{5}{6}\right)^{3} \\
&= \sum_{k=0}^{2}\binom{5}{k}\left(\frac{1}{6}\right)^{k}\left(\frac{5}{6}\right)^{5-k} \\
&= \frac{1}{2}(0.9421 + 0.9734) = 0.9577 \qquad \text{(from binomial table)}
\end{aligned}
$$



Example2

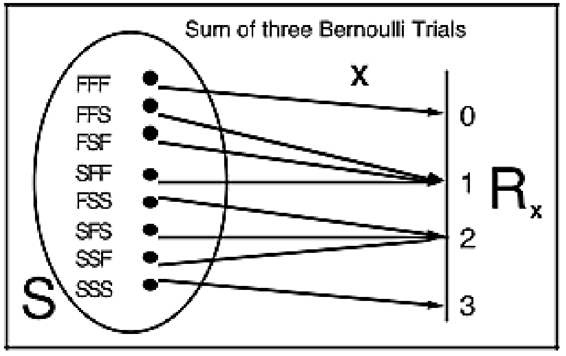
Let X_1, X_2, X_3 be three independent Bernoulli random variables with the same probability of success p. What is the probability density function of the random variable $X = X_1 + X_2 + X_3$? What is the mean and the variance of X?

Sol:

The sample space of the three independent Bernoulli trials is

S = { FFF, FFS, FSF, SFF, FSS, SFS, SSF, SSS } .

The random variable $X = X_1 + X_2 + X_3$ represents the number of successes in each element of S. The following diagram illustrates this.



Example2

Let p be the probability of success. Then

$$f(0) = P(X = 0) = P(FFF) = (1-p)^3$$
$$f(1) = P(X = 1) = P(FFS) + P(FSF) + P(SFF) = 3\,p\,(1-p)^2$$
$$f(2) = P(X = 2) = P(FSS) + P(SFS) + P(SSF) = 3\,p^2\,(1-p)$$
$$f(3) = P(X = 3) = P(SSS) = p^3.$$

Hence

$$f(x) = \binom{3}{x} p^x\,(1-p)^{3-x}, \qquad x = 0,\, 1,\, 2,\, 3.$$

Thus, X~BIN(3,p). In general, if $X_i \sim BER(p)$, then $\sum_{i=1}^{n} X_i \sim BIN(n,p)$ and hence

$$E\left(\sum_{i=1}^{n} X_i\right) = n\,p \quad , \quad Var\left(\sum_{i=1}^{n} X_i\right) = n\,p\,(1-p).$$

Example3

If X~BER(p), What is the p.m.f. of Y=1-X?

Sol:

Since X~BER(p), then $\mathrm{P(x)}=p^{x}(1-p)^{1-x}$. Now, if x=0, then y=1 and if x=1 , then y=0. Also, Y=1-X.

Therefore, $\mathrm{P(y=1\text{-}x)}=p^{1-y}(1-p)^{y}= q^{y}(1-q)^{1-y}$, y =0,1

That is mean: Y=1-X ~BER(q).

Example4

Let X be the number of heads (successes) in n = 7 independent tosses of an unbiased coin. The pmf of X is:

$$p(x) = \begin{cases} \binom{7}{x} \left(\frac{1}{2}\right)^x \left(1 - \frac{1}{2}\right)^{7-x} & x = 0, 1, 2, \ldots, 7 \\ 0 & \text{elsewhere.} \end{cases}$$

Then X has the mgf

$$M(t) = \left(\tfrac{1}{2} + \tfrac{1}{2}e^t\right)^7,$$

has mean $\mu = np = \frac{7}{2}$, and has variance $\sigma^2 = np(1-p) = \frac{7}{4}$. Furthermore, we have

$$P(0 \leq X \leq 1) = \sum_{x=0}^{1} p(x) = \frac{1}{128} + \frac{7}{128} = \frac{8}{128}$$

and

$$P(X = 5) = p(5) = \frac{7!}{5!2!} \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^2 = \frac{21}{128}. \quad \blacksquare$$

Example5

The mgf of a random variable X is $(\frac{2}{3}+\frac{1}{3}e^t)^9$. Show that

$$P(\mu-2\sigma<X<\mu+2\sigma)=\sum_{x=1}^{5}\binom{9}{x}\left(\frac{1}{3}\right)^x\left(\frac{2}{3}\right)^{9-x}.$$

Sol:

Since $n=9$ and $p=1/3$, $\mu=3$ and $\sigma^2=2$. Hence, $\mu-2\sigma=3-2\sqrt{2}$ and $\mu+2\sigma=3+2\sqrt{2}$ and $P(\mu-2\sigma<X<\mu+2\sigma)=P(X=1,2,\ldots,5)$.

Example6

If X ~ BIN(n,p), show that: $E\left(\frac{X}{n}\right) = p$ and $E\left[\left(\frac{X}{n} - p\right)^2\right] = \frac{p(1-p)}{n}$.

Sol:

$$\begin{aligned} E\left(\frac{X}{n}\right) &= \frac{1}{n}E(X) = \frac{1}{n}(np) = p \\ E\left[\left(\frac{X}{n} - p\right)^2\right] &= \frac{1}{n^2}E[(X-np)^2] = \frac{np(1-p)}{n^2} = \frac{p(1-p)}{n}. \end{aligned}$$

Example7

Suppose that X has a Poisson distribution with $\mu = 2$. Compute $P(1 \le X)$

Sol: The pmf of X is

$$p(x) = \begin{cases} \frac{2^x e^{-2}}{x!} & x = 0, 1, 2, \ldots \\ 0 & \text{elsewhere.} \end{cases}$$

Then
$$\begin{aligned} P(1 \le X) &= 1 - P(X = 0) \\ &= 1 - p(0) = 1 - e^{-2} = 0.865, \end{aligned}$$

Example8

If the random variable X has a Poisson distribution such that P(X = 1) = P(X = 2) , find P(X = 4).

Sol:

$$\frac{e^{-\mu}\mu}{1!} = \frac{e^{-\mu}\mu^2}{2!} \Rightarrow \mu = 2 \text{ and } P(X = 4) = \frac{e^{-2}2^4}{4!}.$$

Example9

1-The mgf of a random variable X is $e^{4(e^t-1)}$. Show that

$$P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.931$$

Sol:

Try to solve

2- Let X have a Poisson distribution with mean 1. Compute, if it exists, the expected value E(X!).?