LECTURE NOTE

ON

PROBABILITY AND STATISTICS 2

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LECTURE 7#

➢ Outline :-

- ✓ Continuous distributions

  1- Uniform distribution

Definition

Expected value Variance

Moment generating function

Characteristic function

Distribution function

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Uniform distribution

Definition: A random variable X is said to be uniform on the interval [l,u], if its probability density function is of the form :

$$f(x) = \frac{1}{u-l} \quad , \qquad l \le x \le u$$

where a and b are constants. We denote a random variable X with the uniform distribution on the interval [l, u] as X ~ UNIF(*l*, u).

Theorem: If X is uniform on the interval [l, u] then the mean, variance and moment generating function of X are given by:

$$E(X) = \frac{u+l}{2}, \qquad Var(X) = \frac{(u-l)^2}{12}, M(t) = \frac{1}{(u-l)}[\exp(tu) - \exp(tl)]$$

Proof:

$$\mathbf{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_l^u x \frac{1}{u-l}\,dx = \frac{1}{u-l}\int_l^u x\,dx = \frac{1}{u-l}\left[\frac{1}{2}x^2\right]_l^u$$

$$= \frac{1}{u-l}\frac{1}{2}\left[u^2 - l^2\right] = \frac{(u-l)(u+l)}{2(u-l)} = \frac{u+l}{2}$$

Uniform distribution

Now, we want to find the variance of X:

$$\mathrm{E}\left[X^2\right] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx = \int_l^u x^2 \frac{1}{u-l}\,dx = \frac{1}{u-l}\int_l^u x^2\,dx = \frac{1}{u-l}\left[\frac{1}{3}x^3\right]_l^u = \frac{1}{u-l}\frac{1}{3}\left[u^3 - l^3\right]$$

$$= \frac{(u-l)\left(u^2+ul+l^2\right)}{3(u-l)} = \frac{u^2+ul+l^2}{3}$$

Also,

$$\mathrm{E}[X]^2 = \left(\frac{u+l}{2}\right)^2 = \frac{u^2+2ul+l^2}{4}$$, then

$$\begin{aligned}
\mathrm{Var}[X] &= \mathrm{E}\left[X^2\right] - \mathrm{E}[X]^2 \\
&= \frac{u^2+ul+l^2}{3} - \frac{u^2+2ul+l^2}{4} \\
&= \frac{4u^2+4ul+4l^2-3u^2-6ul-3l^2}{12} \\
&= \frac{(4-3)u^2+(4-6)ul+(4-3)l^2}{12} \\
&= \frac{u^2-2ul+l^2}{12} = \frac{(u-l)^2}{12}
\end{aligned}$$

Using the definition of m.g.f.:

$$\begin{aligned}
M_X(t) &= \mathrm{E}\left[\exp(tX)\right] = \int_{-\infty}^{\infty} \exp(tx) f_X(x)\,dx \\
&= \int_l^u \exp(tx)\frac{1}{u-l}\,dx = \frac{1}{u-l}\left[\frac{1}{t}\exp(tx)\right]_l^u \\
&= \frac{\exp(tu)-\exp(tl)}{(u-l)\,t}
\end{aligned}$$

Uniform distribution

Theorem: The characteristic function of a uniform random variable X is :

$$\varphi_X(t) = \begin{cases} \frac{1}{(u-l)it}\left[\exp(itu) - \exp(itl)\right] & \text{if } t \neq 0 \\ 1 & \text{if } t = 0 \end{cases}$$

Proof: Using the definition of characteristic function:

$$\begin{aligned}
\varphi_X(t) &= \mathrm{E}\left[\exp(itX)\right] = \mathrm{E}\left[\cos(tX)\right] + i\mathrm{E}\left[\sin(tX)\right] \\
&= \int_{-\infty}^{\infty} \cos(tx) f_X(x)\, dx + i \int_{-\infty}^{\infty} \sin(tx) f_X(x)\, dx \\
&= \int_l^u \cos(tx) \frac{1}{u-l} dx + i \int_l^u \sin(tx) \frac{1}{u-l} dx = \frac{1}{u-l}\left\{\int_l^u \cos(tx)\, dx + i \int_l^u \sin(tx)\, dx\right\} \\
&= \frac{1}{u-l}\left\{\left[\frac{1}{t}\sin(tx)\right]_l^u + i\left[-\frac{1}{t}\cos(tx)\right]_l^u\right\} = \frac{1}{(u-l)\,t}\left\{\sin(tu) - \sin(tl) - i\cos(tu) + i\cos(tl)\right\} \\
&= \frac{1}{(u-l)\,it}\left\{i\sin(tu) - i\sin(tl) + \cos(tu) - \cos(tl)\right\} = \frac{1}{(u-l)\,it}\left\{\left[\cos(tu) + i\sin(tu)\right] - \left[\cos(tl) + i\sin(tl)\right]\right\} \\
&= \frac{\exp(itu) - \exp(itl)}{(u-l)\,it}
\end{aligned}$$

Uniform distribution

Theorem: The Distribution function of a uniform random variable X is :

$$F_X(x) = \begin{cases} 0 & \text{if } x < l \\ (x-l)/(u-l) & \text{if } l \le x \le u \\ 1 & \text{if } x > u \end{cases}$$

Proof: If x < l, then $F_X(x) = \mathrm{P}(X \le x) = 0$ because X can not take on values smaller than l. $if\ l \le x \le u,$ then:

$$\begin{aligned} F_X(x) &= \mathrm{P}(X \le x) \\ &= \int_{-\infty}^{x} f_X(t)\,dt \\ &= \int_{l}^{x} \frac{1}{u-l}\,dt \\ &= \frac{1}{u-l}[t]_l^x \\ &= (x-l)/(u-l) \end{aligned}$$

If x > u, then $F_X(x) = \mathrm{P}(X \le x) = 1$

because X can not take on values greater than u.

Uniform distribution

1- Suppose Y ~ UNIF(0, 1) and Y $=\frac{1}{4}X^2$. What is the probability density function of X?

Sol: We shall find the probability density function of X through the cumulative distribution function of Y . The cumulative distribution function of X is given by:

$$F(x) = P(X \le x) = P(X^2 \le x^2) = P\left(\frac{1}{4}X^2 \le \frac{1}{4}x^2\right) = P\left(Y \le \frac{x^2}{4}\right) = \int_0^{\frac{x^2}{4}} f(y)\,dy$$

$$= \int_0^{\frac{x^2}{4}} dy = \frac{x^2}{4}.$$

Thus, $f(x) = \frac{d}{dx}F(x) = \frac{x}{2}$. Hence the probability density function of X is given by:

$$f(x) = \begin{cases} \frac{x}{2} & \text{for } 0 \le x \le 2 \\ 0 & \text{otherwise.} \end{cases}$$

Uniform distribution

2- If X has a uniform distribution on the interval from 0 to 10, then what is

$P\left(X + \frac{10}{X} \geq 7\right)$?

Sol: Since X ~ UNIF(0, 10), the probability density function of X is $f(x) = \frac{1}{10}$ for $0 \leq x \leq 10$.

Hence,

$$P\left(X + \frac{10}{X} \geq 7\right) = P\left(X^2 + 10 \geq 7X\right) = P\left(X^2 - 7X + 10 \geq 0\right) = P\left((X-5)(X-2) \geq 0\right)$$

$$= P\left(X \leq 2 \text{ or } X \geq 5\right) = 1 - P\left(2 \leq X \leq 5\right) = 1 - \int_2^5 f(x)\,dx = 1 - \int_2^5 \frac{1}{10}\,dx$$

$$= 1 - \frac{3}{10} = \frac{7}{10}.$$

3- A box to be constructed so that its height is 10 inches and its base is X inches by X inches. If X has a uniform distribution over the interval (2, 8), then what is the expected volume of the box in cubic inches?

Sol: Since X ~ UNIF(2, 8), $f(x) = \frac{1}{8-2} = \frac{1}{6}$ on $(2, 8)$. The volume V of the box is: $V = 10\,X^2$.

Hence,

$$E(V) = E\left(10X^2\right) = 10\,E\left(X^2\right) = 10\int_2^8 x^2 \frac{1}{6}\,dx = \frac{10}{6}\left[\frac{x^3}{3}\right]_2^8 = \frac{10}{18}\left[8^3 - 2^3\right] = (5)(8)(7) = 280 \text{ cubic inches.}$$

Gamma distribution

The gamma distribution involves the notion of gamma function. First, we develop the notion of gamma function and study some of its well known properties. The gamma function, Γ(z), is a generalization of the notion of factorial. The gamma function is defined as:

$$\Gamma(z) := \int_0^\infty x^{z-1}\, e^{-x}\, dx,$$

where z is positive real number (that is, z > 0).

Lemma 1: Γ(1) = 1.

Proof:

$$\Gamma(1) = \int_0^\infty x^0\, e^{-x}\, dx = \left[-e^{-x}\right]_0^\infty = 1.$$

Lemma 2: The gamma function Γ(z) satisfies the functional equation

$$\Gamma(z) = (z-1)\,\Gamma(z-1) \text{ for all real number } z > 1$$

Proof: Let z be a real number such that z > 1, and consider $\Gamma(z) = \int_0^\infty x^{z-1}\, e^{-x}\, dx$

$$= \left[-x^{z-1}\, e^{-x}\right]_0^\infty + \int_0^\infty (z-1)\, x^{z-2}\, e^{-x}\, dx \quad = (z-1) \int_0^\infty x^{z-2}\, e^{-x}\, dx \quad = (z-1)\,\Gamma(z-1).$$

Gamma distribution

Lemma 3: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Proof: We want to show that $\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx$ is equal to $\sqrt{\pi}$. We substitute $y = \sqrt{x}$, hence the above integral becomes

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty \frac{e^{-x}}{\sqrt{x}}\,dx = 2\int_0^\infty e^{-y^2}\,dy, \qquad \text{where } y = \sqrt{x}.$$

Hence, $\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-u^2}\,du$ and also $\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-v^2}\,dv.$

Multiplying the above two expressions, we get $\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = 4\int_0^\infty \int_0^\infty e^{-(u^2+v^2)}\,du\,dv.$

Now we change the integral into polar form by the transformation:

$u = r\cos(\theta)$ and $v = r\sin(\theta)$, The Jacobian of the transformation is

$$J(r,\theta) = det\begin{pmatrix} \frac{\partial u}{\partial r} & \frac{\partial u}{\partial \theta} \\ \frac{\partial v}{\partial r} & \frac{\partial v}{\partial \theta} \end{pmatrix} = det\begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix} = r\cos^2(\theta) + r\sin^2(\theta) = r.$$

Hence, $\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = 4\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} J(r,\theta)\,dr\,d\theta = 4\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\, r\,dr\,d\theta = 2\int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2}\, 2r\,dr\,d\theta$

Gamma distribution

Lemma 3: $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$

Proof:

$$= 2\int_0^{\frac{\pi}{2}} \int_0^{\infty} e^{-r^2}\, 2r\, dr\, dt = 2\int_0^{\frac{\pi}{2}} \Gamma(1)\, d\theta \;\; = \pi.$$

Note: If n is a natural number, then $\Gamma(n+1) = n!$.

Therefore, we get $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$

Lemma 4 : $\Gamma\left(-\frac{1}{2}\right) = -2\sqrt{\pi}$

Proof: By Lemma 1 , we get: $\Gamma(z) = (z-1)\,\Gamma(z-1)$. Letting $z = \frac{1}{2}$, we get

$\Gamma\left(\frac{1}{2}\right) = \left(\frac{1}{2}-1\right)\Gamma\left(\frac{1}{2}-1\right)$, which is $\Gamma\left(-\frac{1}{2}\right) = -2\,\Gamma\left(\frac{1}{2}\right) = -2\sqrt{\pi}.$

Example: Evaluate $\Gamma\left(\frac{5}{2}\right)$

Answer: $\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\,\frac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\sqrt{\pi}.$

Example: Evaluate $\Gamma\left(-\frac{7}{2}\right)$

Answer: $\Gamma\left(-\frac{1}{2}\right) = -\frac{3}{2}\,\Gamma\left(-\frac{3}{2}\right) = \left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\Gamma\left(-\frac{5}{2}\right)$

$$= \left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right)\Gamma\left(-\frac{7}{2}\right).$$

Hence,

$$\Gamma\left(-\frac{7}{2}\right) = \left(-\frac{2}{3}\right)\left(-\frac{2}{5}\right)\left(-\frac{2}{7}\right)\Gamma\left(-\frac{1}{2}\right) = \frac{16}{105}\sqrt{\pi}.$$

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