LECTURE NOTE

ON

PROBABILITY AND STATISTICS 2

BY

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LECTURE 9#

➢ Outline :-

✓ Continuous distributions

2- Gamma distribution

Definition

Expected value Variance

Moment generating function

Characteristic function

Distribution function

Two special Distributions

Solved exercises

Gamma distribution

Let us take two parameters $\alpha > 0$ and $\beta > 0$. Gamma function $\Gamma(\alpha)$ is defined by:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx. \quad \ldots\ldots\ldots(*)$$

Let $y = \beta x \longrightarrow x = \frac{y}{\beta}$ and then $dx = \frac{1}{\beta} dy$. Then,

If we divide both sides of (*) by $\Gamma(\alpha)$ we get :

$$1 = \int_0^\infty \frac{1}{\Gamma(\alpha)} x^{\alpha-1} e^{-x} dx = \int_0^\infty \frac{1}{\Gamma(\alpha)\beta^\alpha} y^{\alpha-1} e^{-\frac{y}{\beta}} dy \quad \ldots.(**)$$

Then the integration in (**) will be a probability density function since it is nonnegative and it integrates to one.

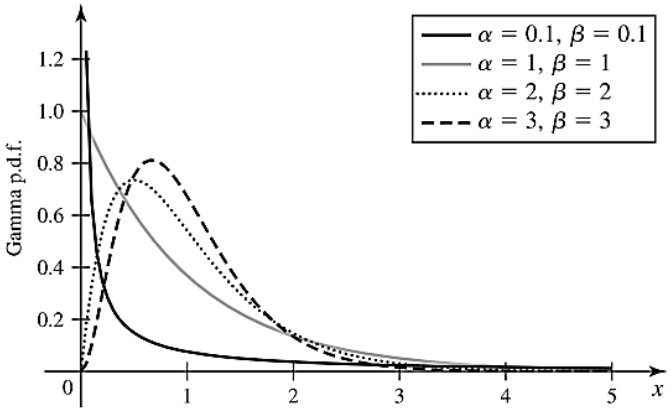
Therefore, we get the following definition:

Gamma distribution

Definition : A continuous random variable X is said to have a gamma distribution if its probability density function is given by:

$$f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\,\theta^{\alpha}}\, x^{\alpha-1}\, e^{-\frac{x}{\theta}} & \text{if} \quad 0 < x < \infty \\ 0 & \text{otherwise,} \end{cases}$$

where α > 0 and θ > 0. We denote a random variable with gamma distribution as X ~ GAM(θ, α). The following diagram shows the graph of the gamma density for various values of values of the parameters θ and α.



Gamma distribution

Theorem: If X ~ GAM(θ, α), then, E(X) = θ α , Var(X) = θ^2 α and

$$M(t) = \left(\frac{1}{1-\theta\, t}\right)^{\alpha}, \qquad \text{if} \qquad t < \frac{1}{\theta}.$$

Proof: First, we derive the moment generating function of X and then we compute the mean and variance of it. The moment generating function:

$$\begin{aligned} M(t) &= E\left(e^{tX}\right) \\ &= \int_0^\infty \frac{1}{\Gamma(\alpha)\,\theta^\alpha}\, x^{\alpha-1}\, e^{-\frac{x}{\theta}}\, e^{tx}\, dx \\ &= \int_0^\infty \frac{1}{\Gamma(\alpha)\,\theta^\alpha}\, x^{\alpha-1}\, e^{-\frac{1}{\theta}(1-\theta t)x}\, dx \\ &= \int_0^\infty \frac{1}{\Gamma(\alpha)\,\theta^\alpha}\, \frac{\theta^\alpha}{(1-\theta t)^\alpha}\, y^{\alpha-1}\, e^{-y}\, dy, \qquad \text{where} \quad y = \frac{1}{\theta}(1-\theta t)x \\ &= \frac{1}{(1-\theta t)^\alpha} \int_0^\infty \frac{1}{\Gamma(\alpha)}\, y^{\alpha-1}\, e^{-y}\, dy \\ &= \frac{1}{(1-\theta t)^\alpha}, \qquad \text{since the integrand is } GAM(1,\alpha). \end{aligned}$$

$$\boxed{x = \frac{\theta\, y}{(1-\theta t)}, \quad dx = \frac{\theta}{(1-\theta t)}\, d\,y}$$

Gamma distribution

The first derivative of the moment generating function is:

$$\begin{aligned} M'(t) &= \frac{d}{dt}(1-\theta t)^{-\alpha} \\ &= (-\alpha)\,(1-\theta t)^{-\alpha-1}\,(-\theta) \\ &= \alpha\,\theta\,(1-\theta t)^{-(\alpha+1)}. \end{aligned}$$

Hence from above, we find the expected value of X to be $E(X) = M'(0) = \alpha\,\theta.$

Similarly,

$$\begin{aligned} M''(t) &= \frac{d}{dt}\left(\alpha\,\theta\,(1-\theta t)^{-(\alpha+1)}\right) \\ &= \alpha\,\theta\,(\alpha+1)\,\theta\,(1-\theta t)^{-(\alpha+2)} \\ &= \alpha\,(\alpha+1)\,\theta^2\,(1-\theta t)^{-(\alpha+2)}. \end{aligned}$$

Thus, the variance of X is

$$Var(X) = M''(0) - (M'(0))^2 = \alpha\,(\alpha+1)\,\theta^2 - \alpha^2\,\theta^2 \;= \alpha\,\theta^2$$

Gamma distribution

Theorem: The characteristic function of a Gamma random variable X is:

$$\varphi(t) = \frac{1}{(1-\theta i t)^{\alpha}}.$$

Proof: By the same procedure for m.g.f.

Distribution function: The distribution function of a Gamma random variable is:

$F(X) = P(X \leq x) = \frac{\Gamma_x(\alpha)}{\Gamma(\alpha)}$, where $\Gamma_x(\alpha)$ is incomplete gamma function and it has the formula:

$$\Gamma_x(\alpha) = \int_0^x y^{\alpha-1} e^{-y} dy$$

Remark: Two special cases of gamma-distributed random variables merit particular consideration.(two special distributions)

Exponential Distribution

Definition: A continuous random variable is said to be an exponential random variable with parameter θ if its probability density function is of the form:

$$\boxed{f(x) = \begin{cases} \frac{1}{\theta} e^{-\frac{x}{\theta}} & \text{if } x > 0 \\ 0 & \text{otherwise,} \end{cases}}$$

, where $\theta > 0$. If a random variable X has an exponential density function with parameter θ, then we denote it by writing $X \sim EXP(\theta)$.

Note: An exponential distribution is a special case of the gamma distribution. If the parameter $\alpha = 1$, then the gamma distribution reduces to the exponential distribution. Hence most of the information about an exponential distribution can be obtained from the gamma distribution.

Example: Let X have the density function :

$$\boxed{f(x) = \begin{cases} \frac{1}{\Gamma(\alpha)\,\theta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\theta}} & \text{if} \quad 0 < x < \infty \\ 0 & \text{otherwise,} \end{cases}}$$

where $\alpha > 0$ and $\theta > 0$. If $\alpha = 4$, what is the mean of $\frac{1}{X^3}$?

Exponential Distribution

Answer:

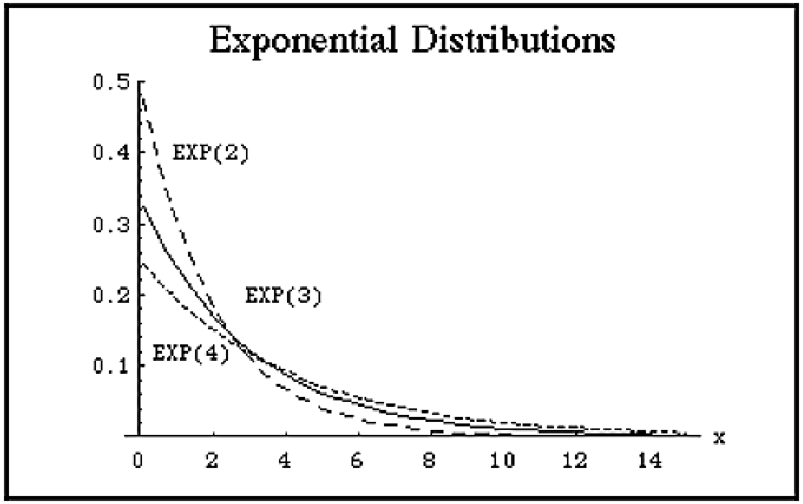
$$E\left(X^{-3}\right) = \int_0^\infty \frac{1}{x^3}\, f(x)\, dx$$

$$= \int_0^\infty \frac{1}{x^3}\, \frac{1}{\Gamma(4)\,\theta^4}\, x^3\, e^{-\frac{x}{\theta}}\, dx$$

$$= \frac{1}{3!\,\theta^4} \int_0^\infty e^{-\frac{x}{\theta}}\, dx$$

$$= \frac{1}{3!\,\theta^3} \int_0^\infty \frac{1}{\theta}\, e^{-\frac{x}{\theta}}\, dx$$

$$= \frac{1}{3!\,\theta^3} \qquad \text{since the integrand is } \mathrm{GAM}(\theta, 1).$$



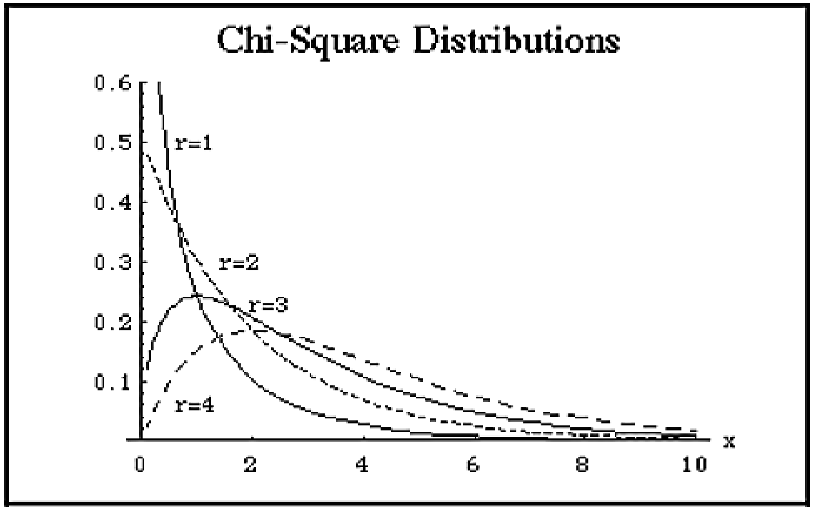
Chi-square Distribution

Definition: A continuous random variable X is said to have a chi-square distribution with r degrees of freedom if its probability density function is of the form:

$$f(x) = \begin{cases} \dfrac{1}{\Gamma\left(\frac{r}{2}\right) 2^{\frac{r}{2}}} x^{\frac{r}{2}-1} e^{-\frac{x}{2}} & \text{if} \quad 0 < x < \infty \\ 0 & \text{otherwise,} \end{cases}$$

where r > 0. If X has a chi-square distribution, then we denote it by writing $X \sim \chi^2(r)$.

Note: The gamma distribution reduces to the chi-square distribution if $\alpha = \frac{r}{2}$ and θ = 2. Thus, the chi-square distribution is a special case of the gamma distribution. Hence most of the information about an chi-square distribution can be obtained from the gamma distribution.



Example: If X ~ GAM(1, 1), then what is the probability density function of the random variable 2X?

Answer: We will use the moment generating method to find the distribution of 2X. The moment generating function of a gamma random variable is given by

$$M(t) = (1 - \theta t)^{-\alpha}, \qquad \text{if} \qquad t < \frac{1}{\theta}.$$

Since X ~ GAM(1, 1), the moment generating function of X is given by :

$$M_X(t) = \frac{1}{1-t}, \qquad t < 1.$$

Hence, the moment generating function of 2X is :

$$M_{2X}(t) = M_X(2t) = \frac{1}{1-2t} \boxed{= \frac{1}{(1-2t)^{\frac{2}{2}}}}$$

The m.g.f. of $\chi^2(2)$.

Hence, if X is an exponential with parameter 1, then 2X is chi-square with 2 degrees of freedom.

SEE YOU IN THE NEXT LECTURE