LECTURE NOTE

ON

PROBABILITY AND STATISTICS 2

BY

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LECTURE 10#

➢ Outline :-

-Solving exercises of Binomial , Poisson and geometric Distribution

1) On a five-question multiple-choice test there are five possible answers, of which one is correct. If a student guesses randomly and independently, what is the probability that she is correct only on two questions?

Solution: Here the probability of success is $\frac{1}{5}$ and thus $1 - p = \frac{4}{5}$. There are $\binom{5}{2}$ different ways she can be correct on two questions. Therefore, the probability that she is correct on two questions is:

$$P(\text{correct on two questions}) = \binom{5}{2} p^2(1-p)^3 = 10\left(\frac{1}{5}\right)^2\left(\frac{4}{5}\right)^3 = \frac{640}{5^5} = 0.2048.$$

2) What is the probability of rolling two sixes and three nonsixes in 5 independent casts of a fair die?

Solution : Let the random variable X denote the number of sixes in 5 independent casts of a fair die. Then X is a binomial random variable with probability of success p and n = 5. The probability of getting a six is $\frac{1}{6}$. Hence:

$$P(X=2) = f(2) = \binom{5}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3 = 10\left(\frac{1}{36}\right)\left(\frac{125}{216}\right) = \frac{1250}{7776} = 0.160751.$$

3) What is the probability of rolling at most two sixes in 5 independent casts of a fair die?

Answer: Let the random variable X denote number of sixes in 5 independent casts of a fair die. Then X is a binomial random variable with probability of success p and n = 5. The probability of getting a six is $\frac{1}{6}$.

Hence, the probability of rolling at most two sixes is :

$$P(X \leq 2) = F(2) = f(0) + f(1) + f(2)$$

$$= \binom{5}{0} \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^5 + \binom{5}{1} \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^4 + \binom{5}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^3$$

$$= \sum_{k=0}^{2} \binom{5}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{5-k}$$

$$= \frac{1}{2}(0.9421 + 0.9734) = 0.9577$$ (from binomial table)

4) Suppose that 2000 points are selected independently and at random from the unit squares $S = \{(x, y) \mid 0 \leq x, y \leq 1\}$. Let X equal the number of points that fall in $A = \{(x, y) \mid x^2 + y^2 < 1\}$. How is X distributed? What are the mean, variance and standard deviation of X?

Answer: If a point falls in A, then it is a success. If a point falls in the complement of A, then it is a failure. The probability of success is

$$p = \frac{\text{area of A}}{\text{area of S}} = \frac{1}{4}\pi.$$

Since, the random variable represents the number of successes in 2000 independent trials, the random variable X is a binomial with parameters $p = \frac{\pi}{4}$ and n = 2000, that is X ~ BIN(2000, $\frac{\pi}{4}$).

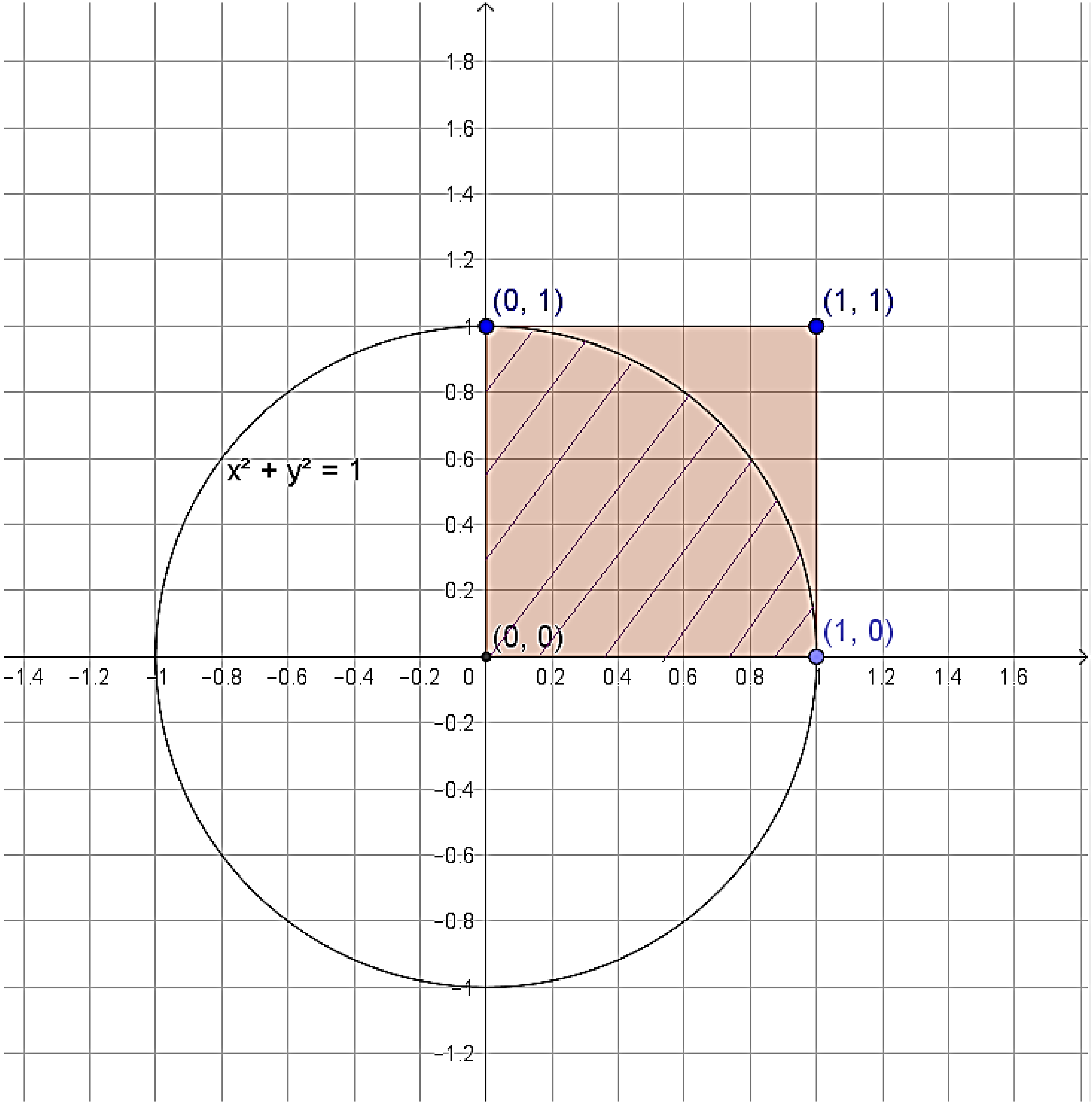
Therefore,

$$\mu_X = 2000\,\frac{\pi}{4} = 1570.8,$$

and

$$\sigma_X^2 = 2000\left(1 - \frac{\pi}{4}\right)\frac{\pi}{4} = 337.1.$$

The standard deviation of X is $\sigma_X = \sqrt{337.1} = 18.36.$


(0, 1)
(1, 1)
(1, 0)
(0, 0)
x² + y² = 1

5) Let the probability that the birth weight (in grams) of babies in America is less than 2547 grams be 0.1. If X equals the number of babies that weigh less than 2547 grams at birth among 20 of these babies selected at random, then what is $P(X \leq 3)$?

Answer: If a baby weighs less than 2547, then it is a success; otherwise it is a failure. Thus X is a binomial random variable with probability of success p and n = 20. We are given that p = 0.1. Hence

$$P(X \leq 3) = \sum_{k=0}^{3} \binom{20}{k} \left(\frac{1}{10}\right)^{k} \left(\frac{9}{10}\right)^{20-k}$$

$$= 0.867 \qquad \text{(from table)}.$$

6- Suppose that on a given weekend the number of accidents at a certain intersection has the Poisson distribution with mean 0.7. What is the probability that there will be at least three accidents at the intersection during the weekend?

Sol: Let X be the number of accidents at a certain intersection. Then X~POI(0.7).
Form the table of the Poisson distribution that given in lecture 3, we get:
$P(X \geq 3)$ =1-P(X<3)=1-[P(X=0)+P(X=1)+P(X=2)]=1-0.9659=0.0341

2- Let X~POI(λ). If P(X=1)=2P(X=2), find λ?

Sol: P(X=1)=2P(X=2) $\longrightarrow$ $\lambda e^{-\lambda} = \frac{2\lambda^2 e^{-\lambda}}{2!}$ $\longrightarrow$ $\lambda(\lambda - 1) = 0$

$\longrightarrow$ λ=1 (λ=0 ignore)

Sampling without Replacement. Suppose that a box contains A red balls and B blue balls. Suppose also that $n \geq 0$ balls are selected at random from the box without replacement, and let X denote the number of red balls that are obtained. Clearly, we must have $n \leq A + B$ or we would run out of balls. Also, if $n = 0$, then $X = 0$ because there are no balls, red or blue, drawn. For cases with $n \geq 1$, we can let $X_i = 1$ if the ith ball drawn is red and $X_i = 0$ if not. Then each X_i has a Bernoulli distribution, but $X_1, \ldots, X_n$ are not independent in general. To see this, assume that both $A > 0$ and $B > 0$ as well as $n \geq 2$. We will now show that $\Pr(X_2 = 1|X_1 = 0) \neq \Pr(X_2 = 1|X_1 = 1)$. If $X_1 = 1$, then when the second ball is drawn there are only $A - 1$ red balls remaining out of a total of $A + B - 1$ available balls. Hence, $\Pr(X_2 = 1|X_1 = 1) = (A - 1)/(A + B - 1)$. By the same reasoning,

$$\Pr(X_2 = 1|X_1 = 0) = \frac{A}{A + B - 1} > \frac{A - 1}{A + B - 1}.$$

Hence, X_2 is not independent of X_1, and we should not expect X to have a binomial distribution. ◀

1- Suppose that Y is a random variable with a geometric distribution. Show that

a $\sum_y p(y) = \sum_{y=1}^{\infty} q^{y-1} p = 1.$

b $\frac{p(y)}{p(y-1)} = q$, for $y = 2, 3, \ldots$. This ratio is less than 1, implying that the geometric probabilities are monotonically decreasing as a function of y. If Y has a geometric distribution, what value of Y is the most likely (has the highest probability)?

Solution : b) $\frac{p(y)}{p(y-1)} = \frac{q^{y-1}p}{q^{y-2}p} = q.$ (because Y ~ GEO(p))

Also, The event Y = 1 has the highest probability for all p, 0 < p < 1, because

P(Y=1)=p(1)=$(1-p)^{1-1}\, p = p.$

2- Suppose that 30% of the applicants for a certain industrial job possess advanced training in computer programming. Applicants are interviewed sequentially and are selected at random from the pool. Find the probability that the first applicant with advanced training in programming is found on the fifth interview.

Solution : Let X be the Applicants are interviewed sequentially and are selected at random from the pool . So X ~ GEO(p=0.3) and then:

P(X=5)=p(5)=$(1-0.3)^{5-1}\, 0.3 = 0.7^4\, 0.3 = 0.07203.$

3- Suppose that X has the geometric distribution with parameter p. Show that for every positive integer a,

$$P(Y > a) = q^a.$$

Solution:

$$P(Y>a)=\sum_{y=a+1}^{\infty} q^{y-1}p = q^a \sum_{x=1}^{\infty} q^{x-1}p = q^a . \quad \text{(because } \sum_{y=a+1}^{\infty} q^{y-1}p = q^a p + q^{a+1}p + \cdots)$$

SEE YOU IN THE NEXT LECTURE