LECTURE NOTE

ON

PROBABILITY AND STATISTICS 2

BY

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LECTURE 11#

➢ Outline :-

✓ Continuous distributions

3- Normal distribution

Definition

Expected value Variance

Moment generating function

Characteristic function

Standard Normal distribution

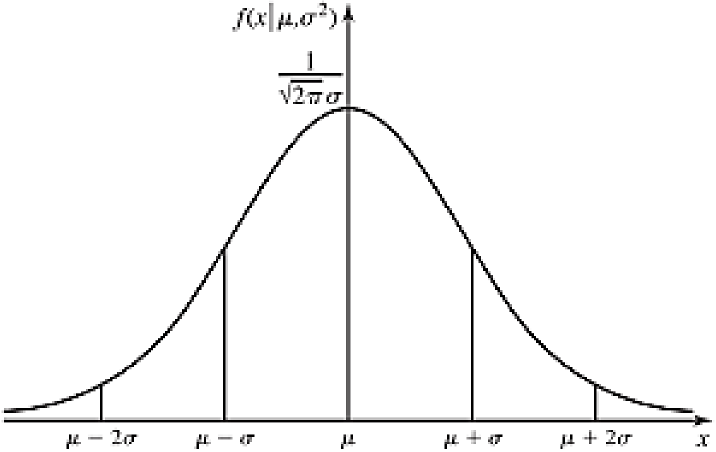
Normal distribution

Definition: A random variable X is said to have a normal distribution if its probability density function is given by:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \qquad -\infty < x < \infty,$$

where $-\infty < \mu < \infty$ and $0 < \sigma^2 < \infty$ are arbitrary parameters. If X has a normal distribution with parameters μ and σ^2, then we write X $\sim N(\mu, \sigma^2)$.

Proof: we must check that f is nonnegative and it integrates to 1.The nonnegative part function is always positive. Hence using property of the gamma function, we show that f integrates to 1 on IR.



Normal distribution

Proof:

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx = 2\int_{\mu}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{2}{\sigma\sqrt{2\pi}} \int_0^{\infty} e^{-z} \frac{\sigma}{\sqrt{2z}}\,dz, \quad \text{where} \quad z = \frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2$$

$$= \frac{1}{\sqrt{\pi}} \int_0^{\infty} \frac{1}{\sqrt{z}} e^{-z}\,dz$$

$x=\sigma\sqrt{2z}+\mu$

⟹ $dx=\frac{\sigma}{\sqrt{2z}}dz$

$$= \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}\right)$$

Definition of Gamma function

$$= \frac{1}{\sqrt{\pi}} \sqrt{\pi} = 1.$$

From lecture 7 Lemma 3

Normal distribution

Theorem: If X ~N(μ, σ^2), then $E(X) = \mu$, $Var(X) = \sigma^2$ and $M(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$.

Proof: We prove this theorem by first computing the moment generating function and finding out the mean and variance of X from it.

$$M(t) = E\left(e^{tX}\right) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\sigma z+\mu)} e^{-\frac{1}{2}z^2} dz, \qquad z = \frac{x-\mu}{\sigma}$$

$$= \frac{e^{t\mu}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2t\sigma z - t^2\sigma^2 + t^2\sigma^2)} dz,$$

$$= e^{t\mu + \frac{1}{2}t^2\sigma^2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z^2 - t\sigma)^2} dz$$

Z~N(t σ,1)

So, M(t)=1

Normal distribution

Proof: The first two derivatives of the m.g.f. of X is:

$M'(t) = \left(\mu + \sigma^2 t\right) \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)$, $M''(t) = \left([\mu + \sigma^2 t]^2 + \sigma^2\right) \exp\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)$

Plugging t = 0 into each of these derivatives yields:

$E(X) = M'(0) = \mu$ and $Var(X) = M''(0) - (M'(0))^2 = \sigma^2$

Characteristic function: If X~ N(μ, σ^2), then

$$\varphi_X(t) = \exp\left(i\mu t - \frac{1}{2}\sigma^2 t^2\right)$$

Proof: Same as the proof of m.g.f

Normal distribution

Example: If X is any random variable with mean μ and variance $\sigma^2 > 0$, then what are the mean and variance of the random variable $Y = \frac{X-\mu}{\sigma}$?

Answer: The mean of the random variable Y is :

$$E(Y) = E\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma} E(X-\mu) = \frac{1}{\sigma}(E(X)-\mu) = \frac{1}{\sigma}(\mu-\mu) = 0.$$

The variance of Y is given by:

$$Var(Y) = Var\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2} Var(X-\mu) = \frac{1}{\sigma} Var(X) = \frac{1}{\sigma^2}\sigma^2 = 1.$$

Hence, if we define a new random variable by taking a random variable and subtracting its mean from it and then dividing the resulting by its standard deviation, then this new random variable will have zero mean and unit variance.

Normal distribution

Definition: A normal random variable is said to be standard normal, if its mean is zero and variance is one. We denote a standard normal random variable X by $X \sim N(0,1)$.

The probability density function of standard normal distribution is the following:

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \qquad -\infty < x < \infty.$$

SEE YOU IN THE NEXT LECTURE