LECTURE NOTE

ON

PROBABILITY AND STATISTICS 2

BY

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LECTURE 12#

➢ Outline :-

✓ Continuous distributions

3- Normal distribution

Distribution function

Solved exercises

Normal distribution

Example: If X is any random variable with mean μ and variance $\sigma^2 > 0$, then what are the mean and variance of the random variable $Y = \frac{X-\mu}{\sigma}$?

Answer: The mean of the random variable Y is :

$$E(Y) = E\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma} E(X-\mu) = \frac{1}{\sigma}(E(X)-\mu) = \frac{1}{\sigma}(\mu-\mu) = 0.$$

The variance of Y is given by:

$$Var(Y) = Var\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2} Var(X-\mu) = \frac{1}{\sigma} Var(X) = \frac{1}{\sigma^2}\sigma^2 = 1.$$

Hence, if we define a new random variable by taking a random variable and subtracting its mean from it and then dividing the resulting by its standard deviation, then this new random variable will have zero mean and unit variance.

Normal distribution

Definition: A normal random variable is said to be standard normal, if its mean is zero and variance is one. We denote a standard normal random variable X by

X ~N(0,1).

The probability density function of standard normal distribution is the following:

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \qquad -\infty < x < \infty.}$$

Distribution function: There is no simple formula for the distribution function $F_X(x)$ of a standard normal random variable X because a closed-form expression for the integral $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$ does not exist; hence, its evaluation requires the use of numerical integration techniques. Probabilities and quantiles for random variables with normal distributions are easily found using any program like Matlab or R or….

Normal distribution

Note :Some values of the distribution function of X are used very frequently and people usually learn them by heart:

$$
\begin{aligned}
&F_X(-2.576) = 0.005 && F_X(2.576) = 0.995 \\
&F_X(-2.326) = 0.01 && F_X(2.326) = 0.99 \\
&F_X(-1.96) = 0.025 && F_X(1.96) = 0.975 \\
&F_X(-1.645) = 0.05 && F_X(1.645) = 0.95
\end{aligned}
$$

Note also that: $F_X(-x) = 1 - F_X(x)$ which is due to the symmetry around 0 of the standard normal density and is often used in calculations.

Table III
Normal Distribution

The following table presents the standard normal distribution. The probabilities tabled are

$$P(X \le x) = \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw.$$

Note that only the probabilities for $x \ge 0$ are tabled. To obtain the probabilities for $x < 0$, use the identity $\Phi(-x) = 1 - \Phi(x)$.

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817

Table III
Normal Distribution

The following table presents the standard normal distribution. The probabilities tabled are

$$P(X \le x) = \Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-w^2/2}\, dw.$$

Note that only the probabilities for $x \ge 0$ are tabled. To obtain the probabilities for $x < 0$, use the identity $\Phi(-x) = 1 - \Phi(x)$.

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998
3.5	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998	.9998

Normal distribution

Therefore, if we know how to compute the values of the distribution function of a standard normal distribution (by table), we also know how to compute the values of the distribution function of a normal distribution with mean $\mu$and variance σ^2.

The following theorem is very important and allows us to find probabilities by using the standard normal table.

Theorem: If X ~ N(μ, σ^2), then the random variable $Z = \frac{X-\mu}{\sigma}$, ~N(0,1)

Proof : We will show that Z is standard normal by finding the probability density function of Z. We compute the probability density of Z by cumulative distribution function method.

X =μ+σ Z , then

$$\begin{aligned} F_X(x) &= \mathrm{P}(X \le x) \\ &= \mathrm{P}(\mu + \sigma Z \le x) \longrightarrow Z = \frac{X-\mu}{\sigma} \longrightarrow X = \mu + \sigma Z \\ &= \mathrm{P}\left(Z \le \frac{x-\mu}{\sigma}\right) \\ &= F_Z\left(\frac{x-\mu}{\sigma}\right) \end{aligned}$$

Hence,

$$f(z) = F'(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2}.$$

Normal distribution

Example: If X ~ N(0, 1), what is the probability of the random variable X less than or equal to − 1.72?

Answer:

$$
\begin{aligned}
P(X \le -1.72) &= 1 - P(X \le 1.72) \\
&= 1 - 0.9573 \qquad \text{(from table)} \\
&= 0.0427.
\end{aligned}
$$

The following example illustrates how to use standard normal table to find probability for normal random variables.

Example: If X ~ N(3, 16), then what is $P(4 \le X \le 8)$?

Answer:

$$P\left(4 \le X \le 8\right) = P\left(\frac{4-3}{4} \le \frac{X-3}{4} \le \frac{8-3}{4}\right) = P\left(\frac{1}{4} \le Z \le \frac{5}{4}\right) = P\left(Z \le 1.25\right) - P\left(Z \le 0.25\right)$$

$$= \underbrace{0.8944 - 0.5987}_{\text{(From table)}} = 0.2957.$$

Normal distribution

Example: If X ~ N(25, 36), then what is the value of the constant c such that

$$P\left(|X - 25| \le c\right) = 0.9544?$$

Answer:

$$0.9544 = P\left(|X - 25| \le c\right) = P\left(-c \le X - 25 \le c\right) = P\left(-\frac{c}{6} \le \frac{X-25}{6} \le \frac{c}{6}\right) = P\left(-\frac{c}{6} \le Z \le \frac{c}{6}\right)$$

$$= P\left(Z \le \frac{c}{6}\right) - P\left(Z \le -\frac{c}{6}\right) = 2\,P\left(Z \le \frac{c}{6}\right) - 1.$$

Hence, $P\left(Z \le \frac{c}{6}\right) = 0.9772$

and from this, using the normal table, we get $\frac{c}{6} = 2$ or $c = 12.$

SEE YOU IN THE NEXT LECTURE