## LECTURE NOTE

## ON

## PROBABILITY AND STATISTICS 2

## BY

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## LECTURE 15\#

> Functions of Random Variables and Their Distribution
1)Distribution Function Method
2) Moment Method for Sums of Random

Variables

## Functions of Random Variables and Their Distribution

In many statistical applications, given the probability distribution of a univariate random variable X , one would like to know the probability distribution of another univariate random variable $\mathrm{Y}=$ $\varphi(\mathrm{X})$, where $\varphi$ is some known function. For example, if we know the probability distribution of the random variable $X$, we would like know the distribution of $Y=\ln (X)$.
For univariate random variable $X$, some commonly used transformed random variable $Y$ of $X$ are: $Y_{n}=X^{2}, Y=|X|, Y=\sqrt{|X|}, Y=\ln (X), Y=\frac{X-\mu}{\sigma}$, and $Y=\left(\frac{X-\mu}{\sigma}\right)^{2}$.

Similarly for a bivariate random variable ( $\mathrm{X}, \mathrm{Y}$ ), some of the most common transformations of X and Y are $\mathrm{X}+\mathrm{Y}, \mathrm{XY}, \frac{X}{Y}, \min \{\mathrm{X}, \mathrm{Y}\}, \max \{\mathrm{X}, \mathrm{Y}\}$ or $\sqrt{X^{2}+Y^{2}}$.

In these lectures, we examine various methods for finding the distribution of a transformed univariate or bivariate random variable, when transformation and distribution of the variable are known. First, we treat the univariate case. Then we treat the bivariate case. We begin with an example for univariate discrete random variable.

## 1)Distribution Function Method

If $Y$ has probability density function $f(y)$ and if $U$ is some function of $Y$, then we can find $F_{U}(u)=P(U \leq u)$ directly by integrating $\mathrm{f}(\mathrm{y})$ over the region for which $U \leq u$. The probability density function for U is found by differentiating $F_{U}(u)$.

The following example illustrates the method.
Example: A box is to be constructed so that the height is 4 inches and its base is X inches by $X$ inches. If $X$ has a standard normal distribution, what is the distribution of the volume of the box?

Answer: The volume of the box is a random variable, since X is a random variable. This random variable V is given by variable. This random variable V is given by $\mathrm{V}=4 \mathrm{X} 2$. To find the density function of $V$, we first determine the form of the distribution function $G(v)$ of V and then we differentiate $\mathrm{G}(\mathrm{v})$ to find the density function of V . The distribution function of V is given by $V=4 X^{2}$.

## 1)Distribution Function Method

$$
\begin{aligned}
G(v) & =P(V \leq v)=P\left(4 X^{2} \leq v\right)=P\left(-\frac{1}{2} \sqrt{v} \leq X \leq \frac{1}{2} \sqrt{v}\right)=\int_{-\frac{1}{2} \sqrt{v}}^{\frac{1}{2} \sqrt{v}} \frac{1}{\sqrt{2 \pi}} e^{-\frac{1}{2} x^{2}} d x \\
& =2 \int_{0}^{\frac{1}{2} \sqrt{v}} \frac{1}{\sqrt{2 \pi}} e^{-\frac{1}{2} x^{2}} d x \quad \text { (since the integrand is even). }
\end{aligned}
$$

Hence, by the Fundamental Theorem of Calculus, we get

$$
\begin{aligned}
g(v) & =\frac{d G(v)}{d v}=\frac{d}{d v}\left(2 \int_{0}^{\frac{1}{2} \sqrt{v}} \frac{1}{\sqrt{2 \pi}} e^{-\frac{1}{2} x^{2}} d x\right)=2 \frac{1}{\sqrt{2 \pi}} e^{-\frac{1}{2}\left(\frac{1}{2} \sqrt{v}\right)^{2}}\left(\frac{1}{2}\right) \frac{d \sqrt{v}}{d v}=\frac{1}{\sqrt{2 \pi}} e^{-\frac{1}{8} v} \frac{1}{2 \sqrt{v}} \\
& =\frac{1}{\Gamma\left(\frac{1}{2}\right) \sqrt{8}} v^{\frac{1}{2}-1} e^{-\frac{v}{8}} \xrightarrow{ } \quad \square \sim G A M\left(8, \frac{1}{2}\right)
\end{aligned}
$$




## 1)Distribution Function Method

Example : If the density function of X is $f(x)= \begin{cases}\frac{1}{2} & \text { for }-1<x<1 \\ 0 & \text { otherwise, }\end{cases}$
what is the probability density function of $Y=X^{2}$ ?
Answer: We first find the cumulative distribution function of Y and then by differentiation, we obtain the density of $Y$. The distribution function $G(y)$ of $Y$ is given by :

$$
G(y)=P(Y \leq y)=P\left(X^{2} \leq y\right)=P(-\sqrt{y} \leq X \leq \sqrt{y})=\int_{-\sqrt{y}}^{\sqrt{y}} \frac{1}{2} d x=\sqrt{y} .
$$

Hence, the density function of Y is given by
$g(y)=\frac{d G(y)}{d y}=\frac{d \sqrt{y}}{d y}=\frac{1}{2 \sqrt{y}}$ for $0<\mathrm{y}<1$.


## 2) Moment Generating Function Method

We know that if X and Y are independent random variables, then

$$
M_{X+Y}(t)=M_{X}(t) M_{Y}(t)
$$

This result can be used to tna the distribution of the sum $X+Y$. Like the convolution method, this method can be used in finding the distribution of $\mathrm{X}+\mathrm{Y}$ if X and Y are independent random variables. We briefly illustrate the method using the following example.
Example: Let $X \sim \operatorname{POI}\left(\lambda_{1}\right)$ and $Y \sim \operatorname{POI}\left(\lambda_{2}\right)$. What is the probability density function of $\mathrm{X}+\mathrm{Y}$ if X and Y are independent?
Answer: Since, $X \sim \operatorname{POI}\left(\lambda_{1}\right)$ and $Y \sim \operatorname{POI}\left(\lambda_{2}\right)$, we get $M_{X}(t)=e^{\lambda_{1}\left(e^{t}-1\right)}$ and

$$
M_{Y}(t)=e^{\lambda_{2}\left(e^{t}-1\right)} .
$$

Further, since X and Y are independent, we have

$$
M_{X+Y}(t)=M_{X}(t) M_{Y}(t)=e^{\lambda_{1}\left(e^{t}-1\right)} e^{\lambda_{2}\left(e^{t}-1\right)}=e^{\lambda_{1}\left(e^{t}-1\right)+\lambda_{2}\left(e^{t}-1\right)}=e^{\left(\lambda_{1}+\lambda_{2}\right)\left(e^{t}-1\right)},
$$

that is, $X+Y \sim \operatorname{POI}\left(\lambda_{1}+\lambda_{2}\right)$. Hence the density function $\mathrm{h}(\mathrm{z})$ of $\mathrm{Z}=\mathrm{X}+\mathrm{Y}$ is given by

$$
h(z)= \begin{cases}\frac{e^{-\left(\lambda_{1}+\lambda_{2}\right)}}{z!}\left(\lambda_{1}+\lambda_{2}\right)^{z} & \text { for } z=0,1,2,3, \ldots \\ 0 & \text { otherwise }\end{cases}
$$

## 2) Moment Generating Function Method

Example: What is the probability density function of the sum of two independent random variable, each of which is gamma with parameters $\theta$ and $\alpha$ ?
Answer: Let X and Y be two independent gamma random variables with parameters $\theta$ and $\alpha$, that is $\mathrm{X} \sim \operatorname{GAM}(\theta, \alpha)$ and $\mathrm{Y} \sim \operatorname{GAM}(\theta, \alpha)$. Therefore: $M_{X}(t)=(1-\theta)^{-\alpha}$ and $M_{Y}(t)=(1-\theta)^{-\alpha}$, respectively. Since, X and Y are independent, we have $M_{X+Y}(t)=M_{X}(t) M_{Y}(t)=(1-\theta)^{-\alpha}(1-\theta)^{-\alpha}=(1-\theta)^{-2 \alpha}$.
Thus $\mathrm{X}+\mathrm{Y}$ has a moment generating function of a gamma random variable with parameters $\theta$ and $2 \alpha$. Therefore $X+Y \sim \operatorname{GAM}(\theta, 2 \alpha)$.
Theorem ( ${ }^{*}$ ) Let $Y_{1}, Y_{2}, \ldots, Y_{s}$ be independent random variables with momentgenerating functions $m_{\gamma_{1}}(t), m_{\gamma_{2}}(t), \ldots, m_{Y_{n}}(t)$, respectively. If $U=Y_{1}+$ $\gamma_{2}+\cdots+Y_{n}$, then

$$
m_{0}(t)=m_{\gamma_{1}}(t) \times m_{V_{2}}(t) \times \cdots \times m_{r_{6}}(t)
$$

Proof:

$$
\begin{aligned}
m_{U}(t) & =E\left[e^{\left.i Y_{1}+\cdots+V_{n}\right)}\right]=E\left(e^{Y_{1}} e^{V_{2}} \cdots e^{Y_{n}}\right) \\
& =E\left(e^{\mu_{1}}\right) \times E\left(e^{\gamma_{2}}\right) \times \cdots \times E\left(e^{\gamma_{r}}\right) .
\end{aligned}
$$

Thus, by the definition of moment-generating functions,

$$
m_{v}(t)=m_{\gamma_{1}}(t) \times m_{r_{2}}(t) \times \cdots \times m_{\gamma_{0}}(t) .
$$

## 2) Moment Generating Function Method

Example: Let $Y_{1}, Y_{2}, \ldots, Y_{n}$ be independent normally distributed random variables with $E\left(Y_{i}\right)=\mu_{i}$ and $V\left(Y_{i}\right)=\sigma_{i}^{2}$, for $i=1,2, \ldots, n$, and let $a_{1}, a_{2}, \ldots, a_{n}$ be constants. If
$U=\sum^{n} a_{i} Y_{i}=a_{1} Y_{1}+a_{2} Y_{2}+\cdots+a_{n} Y_{n}$, then $U$ is a normally distributed random variable with $E(U)=\sum_{i=1}^{n} a_{i} \mu_{i}=a_{1} \mu_{1}+a_{2} \mu_{2}+\cdots+a_{n} \mu_{n}$ and $V(U)=\sum_{i=1}^{n} a_{i}^{2} \sigma_{i}^{2}=a_{1}^{2} \sigma_{1}^{2}+a_{2}^{2} \sigma_{2}^{2}+\cdots+a_{n}^{2} \sigma_{n}^{2}$.
Solution: Because $Y_{i}$ is normally distrihited with mean $\underset{\sigma_{i}^{2} t^{\prime}{ }^{\prime}}{ }$ and variance $\sigma_{i}{ }^{2}, Y_{i}$ has moment-generating function given by $m_{Y_{i}(t)}=\exp \left(\mu_{i} t+\frac{\sigma_{i}^{2} t^{2}}{2}\right)$. Therefore, $a_{i} Y_{i}$ has moment-generating function given by: $m_{a_{i} Y_{i}(t)}=E\left(e^{\operatorname{ta} Y_{i}}\right)=m_{Y_{i}}\left(a_{i} t\right)=\exp \left(\mu_{i} a_{i} t+\frac{a_{i}^{2} \sigma_{i}^{2} t^{2}}{2}\right)$

Because the random variables $Y_{i}$ are independent, the random variables $a_{i} Y_{i}$ are independent, for $\mathrm{i}=1,2, \ldots, \mathrm{n}$, and Theorem $\left(^{*}\right)$ implies that: $m_{U}(t)=m_{a_{1} Y_{1}}(t) \times m_{a_{r_{2}} r_{2}}(t) \times \cdots \times m_{a_{1} r_{n}}(t)$

$$
=\exp \left(\mu_{1} a_{1} t+\frac{a_{1}^{2} \sigma_{1}^{2} t^{2}}{2}\right) \times \cdots \times \exp \left(\mu_{n} a_{n} t+\frac{a_{n}^{2} \sigma_{n}^{2} t^{2}}{2}\right)=\exp \left(t \sum_{i=1}^{n} a_{i} \mu_{i}+\frac{t^{2}}{2} \sum_{i=1}^{n} a_{i}^{2} \sigma_{i}^{2}\right) .
$$

Thus, $U$ has a normal distribution with mean $\sum_{i=1}^{n} a_{i} \mu_{i}$ and variance $\sum_{i=1}^{n} a_{i}^{2} \sigma_{i}^{2}$.

## SEE YOU IN THE NEXT LECTURE

