LECTURE NOTE

ON

PROBABILITY AND STATISTICS 2

BY

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LECTURE 16#

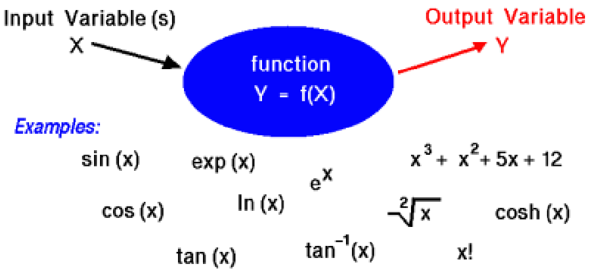
➢ Outline :-

➢ Functions of Random Variables and Their Distribution

3) Transformation Method

- For Univariate Case
- For Bivariate Case
- Different Examples

Functions of Random Variables and Their Distribution



3)Transformation Method for Univariate Case :

Theorem : Let X be a continuous random variable with probability density function f(x). Let y = T(x) be an increasing (or decreasing) function. Then the density function of the random variable Y = T(X) is given by:

$$\boxed{g(y) = \left|\frac{dx}{dy}\right| f(W(y))}$$

where x = W(y) is the inverse function of T(x).

Proof: Suppose y = T(x) is an increasing function. The distribution function G(y) of Y is given by

$$\begin{aligned} G(y) &= P(Y \le y) \\ &= P(T(X) \le y) \\ &= P(X \le W(y)) \\ &= \int_{-\infty}^{W(y)} f(x)\, dx. \end{aligned}$$

3)Transformation Method

Then, differentiating we get the density function of Y , which is:

$$g(y) = \frac{dG(y)}{dy} = \frac{d}{dy}\left(\int_{-\infty}^{W(y)} f(x)\,dx\right) = f(W(y))\,\frac{dW(y)}{dy} = f(W(y))\,\frac{dx}{dy} \qquad (\text{since} \quad x = W(y)).$$

On the other hand, if y = T(x) is a decreasing function, then the distribution function of Y is given by:

$$\begin{aligned} G(y) &= P(Y \le y) = P(T(X) \le y) = P(X \ge W(y)) \qquad (\text{since } T(x) \text{ is decreasing}) \\ &= 1 - P(X \le W(y)) = 1 - \int_{-\infty}^{W(y)} f(x)\,dx. \end{aligned}$$

As before, differentiating we get the density function of Y , which is:

$$g(y) = \frac{dG(y)}{dy} = \frac{d}{dy}\left(1 - \int_{-\infty}^{W(y)} f(x)\,dx\right) = -f(W(y))\,\frac{dW(y)}{dy} = -f(W(y))\,\frac{dx}{dy} \qquad (\text{since} \quad x = W(y)).$$

Hence, combining both the cases, we get:

$$\boxed{g(y) = \left|\frac{dx}{dy}\right| f(W(y))}$$

3)Transformation Method

Example: Let $Z = \frac{X-\mu}{\sigma}$. If $X \sim N\left(\mu, \sigma^2\right)$, what is the probability density function of Z?

Answer: $z = U(x) = \dfrac{x-\mu}{\sigma}$. Hence, the inverse of U is given by:

$$W(z) = x = \sigma z + \mu.$$

Therefore:

$$\frac{dx}{dz} = \sigma.$$

Hence, by above Theorem, the density of Z is given by

$$g(z) = \left|\frac{dx}{dz}\right| f(W(y)) = \sigma \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{W(z)-\mu}{\sigma}\right)^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{z\sigma+\mu-\mu}{\sigma}\right)^2} = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}.$$

Example:

Let $Z = \frac{X-\mu}{\sigma}$. If $X \sim N\left(\mu, \sigma^2\right)$, then show that Z^2 is chi-square with one degree of freedom, that $Z^2 \sim \chi^2(1)$.

Answer:

$$y = T(x) = \left(\frac{x-\mu}{\sigma}\right)^2 \cdot \quad x = \mu + \sigma\sqrt{y}. \quad W(y) = \mu + \sigma\sqrt{y}, \qquad y > 0.$$

$$\frac{dx}{dy} = \frac{\sigma}{2\sqrt{y}}.$$

3)Transformation Method

The density of Y is:

$$g(y) = \left|\frac{dx}{dy}\right| f(W(y)) = \sigma \frac{1}{2\sqrt{y}} f(W(y)) = \sigma \frac{1}{2\sqrt{y}} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{W(y)-\mu}{\sigma}\right)^2} = \frac{1}{2\sqrt{2\pi y}} e^{-\frac{1}{2}\left(\frac{\sqrt{y}\sigma+\mu-\mu}{\sigma}\right)^2}$$

$$= \frac{1}{2\sqrt{2\pi y}} e^{-\frac{1}{2}y} = \frac{1}{2\sqrt{\pi}\sqrt{2}} y^{-\frac{1}{2}} e^{-\frac{1}{2}y} = \frac{1}{2\Gamma\left(\frac{1}{2}\right)\sqrt{2}} y^{-\frac{1}{2}} e^{-\frac{1}{2}y}.$$

Hence $Y \sim \chi^2(1)$.

Example : Let Y = − ln X. If X ~ UNIF(0, 1), then what is the density function of Y where nonzero?

Answer: We are given that : $y = T(x) = -\ln x$.

Hence, the inverse of y = T(x) is given by $W(y) = x \quad = e^{-y}$. Therefore

$$\frac{dx}{dy} = -e^{-y}.$$

Hence, by above Theorem, the probability density of Y is given by

$$g(y) = \left|\frac{dx}{dy}\right| f(W(y)) = e^{-y} f(W(y)) = e^{-y}.$$

Thus Y ~ EXP(1). Hence, if X ~ UNIF(0, 1), then the random variable − ln X ~ EXP(1).

3) Transformation Method for Bivariate Case

Although all the examples we have in this section involve continuous random variables, the transformation method also works for the discrete random variables.

Theorem : Let X and Y be two continuous random variables with joint density f(x, y). Let U = P(X, Y) and V = Q(X, Y) be functions of X and Y . If the functions P(x, y) and Q(x, y) have single valued inverses, say X = R(U, V) and Y = S(U, V), then the joint density g(u, v) of U and V is given by:

$$g(u,v) = |J| \; f(R(u,v), \, S(u,v)),$$

where J denotes the Jacobian and given by $$J = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}.$$

Example: Let X and Y have the joint probability density function

$$f(x,y) = \begin{cases} 8\,xy & \text{for } 0 < x < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

What is the joint density of $U = \frac{X}{Y}$ and V = Y ?

3) Transformation Method for Bivariate Case

Answer: Since $U = \frac{X}{Y}$ and $V = Y$,we get by solving for X and Y:

X = U Y = U V and Y = V.

Hence, the Jacobian of the transformation is given by

$$J = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = v \cdot 1 - u \cdot 0 = v.$$

The joint density function of U and V is

$$g(u,v) = |J|\, f(R(u,v),\, S(u,v)) = |v|\, f(uv, v) = v\, 8\, (uv)\, v = 8\, uv^3.$$

Note that, since 0 < x < y < 1, we have 0 < *uv* < *v* < 1. The last inequalities yield 0 < uv < v and 0 < v < 1. Therefore, we get 0 < u < 1 and 0 < v < 1. Thus, the joint density of U and V is given by

$$g(u,v) = \begin{cases} 8\, uv^3 & \text{for } 0 < u < 1;\ 0 < v < 1 \\ 0 & \text{otherwise.} \end{cases}$$

SEE YOU IN THE NEXT LECTURE