LECTURE NOTE

ON

PROBABILITY AND STATISTICS 2

BY

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LECTURE 17#

➢ Outline :-

- ➢ Functions of Random Variables and Their Distribution

  - More applications

Functions of Random Variables and Their Distribution

Example : Let each of the independent random variables X and Y have the density function:

$$f(x) = \begin{cases} e^{-x} & \text{for } 0 < x < \infty \\ 0 & \text{otherwise.} \end{cases}$$

What is the joint density of U = X and V = 2X + 3Y and the domain on which this density is positive?

Solution: Since U = X , V = 2X + 3Y, we get by solving for X and Y :

$$X = U \quad , \quad Y = \frac{1}{3}\,V - \frac{2}{3}\,U.$$

Hence, the Jacobian of the transformation is given by :

$$J = \frac{\partial x}{\partial u}\,\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\,\frac{\partial y}{\partial u} \quad = 1 \cdot \left(\frac{1}{3}\right) - 0 \cdot \left(-\frac{2}{3}\right) = \frac{1}{3}.$$

Functions of Random Variables and Their Distribution

The joint density function of U and V is:

$$g(u,v) = |J|\ f(R(u,v),\ S(u,v)) = \left|\frac{1}{3}\right|\ f\left(u,\ \frac{1}{3}v - \frac{2}{3}u\right) = \frac{1}{3}e^{-u}\ e^{-\frac{1}{3}v+\frac{2}{3}u} = \frac{1}{3}\ e^{-\left(\frac{u+v}{3}\right)}.$$

Since $0 < x < \infty$, $0 < y < \infty$, we get $0 < u < \infty$,$0 < v < \infty$,

Further, since $v = 2u + 3y$ and $3y > 0$, we have $v > 2u$.

Hence, the domain of g(u, v) where nonzero is given by $0 < 2u < v < \infty$.

The joint density g(u, v) of the random variables U and V is given by:

$$g(u,v) = \begin{cases} \frac{1}{3}\ e^{-\left(\frac{u+v}{3}\right)} & \text{for } 0 < 2u < v < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Example: Let X and Y be independent random variables, each with density function

$$f(x) = \begin{cases} \lambda\, e^{-\lambda x} & \text{for } 0 < x < \infty \\ 0 & \text{otherwise,} \end{cases}$$

where $\lambda > 0$. Let $U = X + 2Y$ and $V = 2X + Y$. What is the joint density of U and V?

Functions of Random Variables and Their Distribution

Answer: Since U = X + 2Y, V = 2X + Y, we get by solving for X and Y:

$$X = -\frac{1}{3}U + \frac{2}{3}V \quad , \quad Y = \frac{2}{3}U - \frac{1}{3}V.$$

Hence, the Jacobian of the transformation is given by:

$$J = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right) - \left(\frac{2}{3}\right)\left(\frac{2}{3}\right) = \frac{1}{9} - \frac{4}{9} = -\frac{1}{3}.$$

The joint density function of U and V is:

$$g(u,v) = |J|\; f(R(u,v),\, S(u,v)) = \left|-\frac{1}{3}\right| f(R(u,v))\; f(S(u,v)) = \frac{1}{3}\lambda\, e^{\lambda R(u,v)}\,\lambda\, e^{\lambda S(u,v)} = \frac{1}{3}\lambda^2\, e^{\lambda[R(u,v)+S(u,v)]}$$

$$= \frac{1}{3}\lambda^2\, e^{-\lambda\left(\frac{u+v}{3}\right)}.$$

Hence, the joint density g(u, v) of the random variables U and V is given by

$$g(u,v) = \begin{cases} \frac{1}{3}\lambda^2\, e^{-\lambda\left(\frac{u+v}{3}\right)} & \text{for } 0 < u < \infty;\; 0 < v < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Functions of Random Variables and Their Distribution

Example: Let X and Y be independent random variables, each with density function:

$$f(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}x^2}, \qquad -\infty < x < \infty.$$

Let $U = \frac{X}{Y}$ and $V = Y$. What is the joint density of U and V? Also, what is the density of U?

Answer: Since $U = \dfrac{X}{Y}$, $V = Y,$ we get by solving for X and Y : X = UV, Y = V.

Hence, the Jacobian of the transformation is given by:

$$J = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = v \cdot (1) - u \cdot (0) \quad = v.$$

The joint density function of U and V is

$$g(u,v) = |J|\ f(R(u,v),\, S(u,v)) = |v|\ f(R(u,v))\ f(S(u,v)) = |v|\ \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}R^2(u,v)}\ \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}S^2(u,v)}$$

$$= |v|\ \frac{1}{2\pi}\, e^{-\frac{1}{2}\left[R^2(u,v)+S^2(u,v)\right]} = |v|\ \frac{1}{2\pi}\, e^{-\frac{1}{2}\left[u^2v^2+v^2\right]} = |v|\ \frac{1}{2\pi}\, e^{-\frac{1}{2}v^2\left(u^2+1\right)}.$$

Hence, the joint density g(u, v) of the random variables U and V is given by

$g(u,v) = |v|\ \dfrac{1}{2\pi}\, e^{-\frac{1}{2}v^2\left(u^2+1\right)},$ where $-\infty < u < \infty$ and $-\infty < v < \infty$.

Functions of Random Variables and Their Distribution

Next, we want to find the density of U. We can obtain this by finding the marginal of U from the joint density of U and V . Hence, the marginal $g_1(u)$ of U is given by

$$g_1(u) = \int_{-\infty}^{\infty} g(u,v)\,dv \quad = \int_{-\infty}^{\infty} |v|\,\frac{1}{2\pi}\,e^{-\frac{1}{2}v^2\left(u^2+1\right)}\,dv$$

$$= \int_{-\infty}^{0} -v\,\frac{1}{2\pi}\,e^{-\frac{1}{2}v^2\left(u^2+1\right)}\,dv + \int_{0}^{\infty} v\,\frac{1}{2\pi}\,e^{-\frac{1}{2}v^2\left(u^2+1\right)}\,dv$$

$$= \frac{1}{2\pi}\,\left(\frac{1}{2}\right)\,\left[\frac{2}{u^2+1}\,e^{-\frac{1}{2}\,v^2\left(u^2+1\right)}\right]_{-\infty}^{0} \quad + \frac{1}{2\pi}\,\left(\frac{1}{2}\right)\,\left[\frac{-2}{u^2+1}\,e^{-\frac{1}{2}\,v^2\left(u^2+1\right)}\right]_{0}^{\infty}$$

$$= \frac{1}{2\pi}\,\frac{1}{u^2+1} + \frac{1}{2\pi}\,\frac{1}{u^2+1} \quad = \frac{1}{\pi\left(u^2+1\right)}.$$