CHAPTER 3

INNER PRODUCT SPACES. HILBERT SPACES

In a normed space we can add vectors and multiply vectors by scalars, just as in elementary vector algebra. Furthermore, the norm on such a space generalizes the elementary concept of the length of a vector. However, what is still missing in a general normed space, and what we would like to have if possible, is an analogue of the familiar dot product

$$a \cdot b = \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3$$

and resulting formulas, notably

$$|a| = \sqrt{a \cdot a}$$

and the condition for orthogonality (perpendicularity)

$$a \cdot b = 0$$

which are important tools in many applications. Hence the question arises whether the dot product and orthogonality can be generalized to arbitrary vector spaces. In fact, this can be done and leads to *inner product spaces* and complete inner product spaces, called *Hilbert spaces.*

Inner product spaces are special normed spaces, as we shall see. Historically they are older than general normed spaces. Their theory is richer and retains many features of Euclidean space, a central concept being orthogonality. In fact, inner product spaces are probably the most natural generalization of Euclidean space, and the reader should note the great harmony and beauty of the concepts and proofs in this field. The whole theory was initiated by the work of D. Hilbert (1912) on integral equations. The currently used geometrical notation and terminology is analogous to that of Euclidean geometry and was coined by E. Schmidt (1908), who followed a suggestion of G. Kowalewski (as he mentioned on p. 56 of his paper). These spaces have been, up to now, the most useful spaces in practical applications of functional analysis.

Important concepts, brief orientation about main content

An *inner product space* X (Def. 3.1-1) is a vector space with an *inner product* $\langle x, y\rangle$ defined on it. The latter generalizes the dot product of vectors in three dimensional space and is used to define

(I) a *norm* $\|\cdot\|$ by $\|x\|=\langle x, x\rangle^{1/2}$,

(II) *orthogonality* by $\langle x, y\rangle=0$.

A *Hilbert space* H is a complete inner product space. The theory of inner product and Hilbert spaces is richer than that of general normed and Banach spaces. Distinguishing features are

(i) representations of H as a direct sum of a closed subspace and its *orthogonal complement* (cf. 3.3-4),

(ii) *orthonormal sets and sequences* and corresponding representations of elements of H (cf. Secs. 3.4, 3.5),

(iii) the *Riesz representation* 3.8-1 of bounded linear functionals by inner products,

(iv) the *Hilbert-adjoint operator* T^* of a bounded linear operator T (cf. 3.9-1).

Orthonormal sets and sequences are truly interesting only if they are total (Sec. 3.6). Hilbert-adjoint operators can be used to define classes of operators (*self-adjoint, unitary, normal;* cf. Sec. 3.10) which are of great importance in applications.

3.1 Inner Product Space. Hilbert Space

The spaces to be considered in this chapter are defined as follows.

3.1-1 Definition (Inner product space, Hilbert space). An *inner product space* (or *pre-Hilbert space*) is a vector space X with an inner product defined on X. A *Hilbert space* is a complete inner product space (complete in the metric defined by the inner product; cf. (2), below). Here, an **inner product** *on* X is a mapping of $X\times X$ into the scalar field K of X; that is, with every pair of vectors x and y there is associated a scalar which is written

$$\langle x, y\rangle$$

and is called the *inner product*[1] of x and y, such that for all vectors x, y, z and scalars α we have

(IP1) $$\langle x+y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$$

(IP2) $$\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$$

(IP3) $$\langle x, y\rangle = \overline{\langle y, x\rangle}$$

(IP4) $$\langle x, x\rangle \geqq 0$$
$$\langle x, x\rangle = 0 \iff x = 0.$$

An inner product on X defines a *norm* on X given by

(1) $$\|x\| = \sqrt{\langle x, x\rangle} \qquad (\geqq 0)$$

and a *metric* on X given by

(2) $$d(x, y) = \|x-y\| = \sqrt{\langle x-y, x-y\rangle}.$$ ∎

Hence *inner product spaces are normed spaces, and Hilbert spaces are Banach spaces.*

In (IP3), the bar denotes complex conjugation. Consequently, if X is a *real* vector space, we simply have

$$\langle x, y\rangle = \langle y, x\rangle \qquad (\textit{Symmetry}).$$

The proof that (1) satisfies the axioms (N1) to (N4) of a norm (cf. Sec. 2.2) will be given at the beginning of the next section.

From (IP1) to (IP3) we obtain the formula

(3)

(a) $$\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$$

(b) $$\langle x, \alpha y\rangle = \bar{\alpha}\langle x, y\rangle$$

(c) $$\langle x, \alpha y + \beta z\rangle = \bar{\alpha}\langle x, y\rangle + \bar{\beta}\langle x, z\rangle$$

[1] Or *scalar product*, but this must not be confused with the product of a vector by a scalar in a vector space.

The notation $\langle\ ,\ \rangle$ for the inner product is quite common. In an elementary text such as the present one it may have the advantage over another popular notation, $(\ ,\ )$, that it excludes confusion with ordered pairs (components of a vector, elements of a product space, arguments of functions depending on two variables, etc.).

which we shall use quite often. (3a) shows that the inner product is linear in the first factor. Since in (3c) we have complex conjugates $\bar{\alpha}$ and $\bar{\beta}$ on the right, we say that the inner product is *conjugate linear* in the second factor. Expressing both properties together, we say that the inner product is *sesquilinear*. This means "$1\frac{1}{2}$ times linear" and is motivated by the fact that "conjugate linear" is also known as "semilinear" (meaning "halflinear"), a less suggestive term which we shall not use.

The reader may show by a simple straightforward calculation that a norm on an inner product space satisfies the important **parallelogram equality**

$$\|x+y\|^2+\|x-y\|^2=2(\|x\|^2+\|y\|^2). \tag{4}$$

This name is suggested by elementary geometry, as we see from Fig. 23 if we remember that the norm generalizes the elementary concept of the length of a vector (cf. Sec. 2.2). It is quite remarkable that such an equation continues to hold in our present much more general setting.

We conclude that if a norm does not satisfy (4), it cannot be obtained from an inner product by the use of (1). Such norms do exist; examples will be given below. Without risking misunderstandings we may thus say:

Not all normed spaces are inner product spaces.

Before we consider examples, let us define the concept of orthogonality, which is basic in the whole theory. We know that if the dot product of two vectors in three dimensional spaces is zero, the vectors are orthogonal, that is, they are perpendicular or at least one of them is the zero vector. This suggests and motivates the following

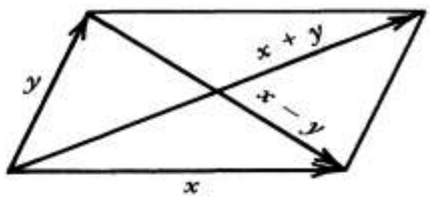

Fig. 23. Parallelogram with sides x and y in the plane

3.1-2 Definition (Orthogonality). An element x of an inner product space X is said to be *orthogonal* to an element $y \in X$ if

$$\langle x, y\rangle = 0.$$

We also say that *x and y are orthogonal*, and we write $x \perp y$. Similarly, for subsets $A, B \subset X$ we write $x \perp A$ if $x \perp a$ for all $a \in A$, and $A \perp B$ if $a \perp b$ for all $a \in A$ and all $b \in B$. ∎

Examples

3.1-3 Euclidean space $\mathbf{R}^n$. The space $\mathbf{R}^n$ is a Hilbert space with inner product defined by

$$\langle x, y\rangle = \xi_1\eta_1 + \cdots + \xi_n\eta_n \tag{5}$$

where $x = (\xi_j) = (\xi_1, \cdots, \xi_n)$ and $y = (\eta_j) = (\eta_1, \cdots, \eta_n)$.

In fact, from (5) we obtain

$$\|x\| = \langle x, x\rangle^{1/2} = (\xi_1{}^2 + \cdots + \xi_n)^{1/2}$$

and from this the Euclidean metric defined by

$$d(x, y) = \|x - y\| = \langle x - y, x - y\rangle^{1/2} = [(\xi_1 - \eta_1)^2 + \cdots + (\xi_n - \eta_n)^2]^{1/2};$$

cf. 2.2-2. Completeness was shown in 1.5-1.

If $n = 3$, formula (5) gives the usual dot product

$$\langle x, y\rangle = x \cdot y = \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3$$

of $x = (\xi_1, \xi_2, \xi_3)$ and $y = (\eta_1, \eta_2, \eta_3)$, and the orthogonality

$$\langle x, y\rangle = x \cdot y = 0$$

agrees with the elementary concept of perpendicularity.

3.1-4 Unitary space $\mathbf{C}^n$. The space $\mathbf{C}^n$ defined in 2.2-2 is a Hilbert space with inner product given by

$$\langle x, y\rangle = \xi_1\bar{\eta}_1 + \cdots + \xi_n\bar{\eta}_n. \tag{6}$$

In fact, from (6) we obtain the norm defined by

$$\|x\|=(\xi_1\bar{\xi}_1+\cdots+\xi_n\bar{\xi}_n)^{1/2}=(|\xi_1|^2+\cdots+|\xi_n|^2)^{1/2}.$$

Here we also see why we have to take complex conjugates $\bar{\eta}_j$ in (6); this entails $\langle y, x\rangle=\overline{\langle x, y\rangle}$, which is (IP3), so that $\langle x, x\rangle$ is real.

3.1-5 Space $L^2[a, b]$. The norm in Example 2.2-7 is defined by

$$\|x\|=\left(\int_a^b x(t)^2\,dt\right)^{1/2}$$

and can be obtained from the inner product defined by

$$\langle x, y\rangle=\int_a^b x(t)y(t)\,dt. \tag{7}$$

In Example 2.2-7 the functions were assumed to be real-valued, for simplicity. In connection with certain applications it is advantageous to remove that restriction and consider *complex-valued* functions (keeping $t\in[a, b]$ real, as before). These functions form a complex vector space, which becomes an inner product space if we define

$$\langle x, y\rangle=\int_a^b x(t)\overline{y(t)}\,dt. \tag{7*}$$

Here the bar denotes the complex conjugate. It has the effect that (IP3) holds, so that $\langle x, x\rangle$ is still real. This property is again needed in connection with the norm, which is now defined by

$$\|x\|=\left(\int_a^b |x(t)|^2\,dt\right)^{1/2}$$

because $x(t)\overline{x(t)}=|x(t)|^2$.

The completion of the metric space corresponding to (7) is the real space $L^2[a, b]$; cf. 2.2-7. Similarly, the completion of the metric space corresponding to (7*) is called the *complex space* $L^2[a, b]$. We shall see in the next section that the inner product can be extended from an inner product space to its completion. Together with our present discussion this implies that $L^2[a, b]$ is a Hilbert space.

3.1-6 Hilbert sequence space l^2. The space l^2 (cf. 2.2-3) is a Hilbert space with inner product defined by

$$\langle x, y\rangle = \sum_{j=1}^{\infty} \xi_j \bar{\eta}_j. \tag{8}$$

Convergence of this series follows from the Cauchy-Schwarz inequality (11), Sec. 1.2, and the fact that $x, y \in l^2$, by assumption. We see that (8) generalizes (6). The norm is defined by

$$\|x\| = \langle x, x\rangle^{1/2} = \left(\sum_{j=1}^{\infty} |\xi_j|^2\right)^{1/2}.$$

Completeness was shown in 1.5-4.

l^2 is the prototype of a Hilbert space. It was introduced and investigated by D. Hilbert (1912) in his work on integral equations. An axiomatic definition of Hilbert space was not given until much later, by J. von Neumann (1927), pp. 15–17, in a paper on the mathematical foundation of quantum mechanics. Cf. also J. von Neumann (1929–30), pp. 63–66, and M. H. Stone (1932), pp. 3–4. That definition included separability, a condition which was later dropped from the definition when H. Löwig (1934), F. Rellich (1934) and F. Riesz (1934) showed that for most parts of the theory that condition was an unnecessary restriction. (These papers are listed in Appendix 3.)

3.1-7 Space l^p. *The space l^p with $p \neq 2$ is not an inner product space, hence not a Hilbert space.*

Proof. Our statement means that the norm of l^p with $p \neq 2$ cannot be obtained from an inner product. We prove this by showing that the norm does not satisfy the parallelogram equality (4). In fact, let us take $x = (1, 1, 0, 0, \cdots) \in l^p$ and $y = (1, -1, 0, 0, \cdots) \in l^p$ and calculate

$$\|x\| = \|y\| = 2^{1/p}, \qquad \|x+y\| = \|x-y\| = 2.$$

We now see that (4) is not satisfied if $p \neq 2$.

l^p is complete (cf. 1.5-4). Hence *l^p with $p \neq 2$ is a Banach space which is not a Hilbert space.* The same holds for the space in the next example.

3.1-8 Space $C[a, b]$. *The space $C[a, b]$ is not an inner product space, hence not a Hilbert space.*

Proof. We show that the norm defined by

$$\|x\| = \max_{t \in J} |x(t)| \qquad\qquad J = [a, b]$$

cannot be obtained from an inner product since this norm does not satisfy the parallelogram equality (4). Indeed, if we take $x(t) = 1$ and $y(t) = (t-a)/(b-a)$, we have $\|x\| = 1$, $\|y\| = 1$ and

$$x(t) + y(t) = 1 + \frac{t-a}{b-a}$$

$$x(t) - y(t) = 1 - \frac{t-a}{b-a}.$$

Hence $\|x+y\| = 2$, $\|x-y\| = 1$ and

$$\|x+y\|^2 + \|x-y\|^2 = 5 \qquad \text{but} \qquad 2(\|x\|^2 + \|y\|^2) = 4.$$

This completes the proof. ∎

We finally mention the following interesting fact. We know that to an inner product there corresponds a norm which is given by (1). It is remarkable that, conversely, we can "rediscover" the inner product from the corresponding norm. In fact, the reader may verify by straightforward calculation that for a real inner product space we have

$$\langle x, y\rangle = \tfrac{1}{4}(\|x+y\|^2 - \|x-y\|^2) \tag{9}$$

and for a complex inner product space we have

$$\begin{aligned} \operatorname{Re}\langle x, y\rangle &= \tfrac{1}{4}(\|x+y\|^2 - \|x-y\|^2) \\ \operatorname{Im}\langle x, y\rangle &= \tfrac{1}{4}(\|x+iy\|^2 - \|x-iy\|^2). \end{aligned} \tag{10}$$

Formula (10) is sometimes called the **polarization identity.**

Problems

1. Prove (4).

2. (Pythagorean theorem) If $x \perp y$ in an inner product space X, show that (Fig. 24)

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

Extend the formula to m mutually orthogonal vectors.

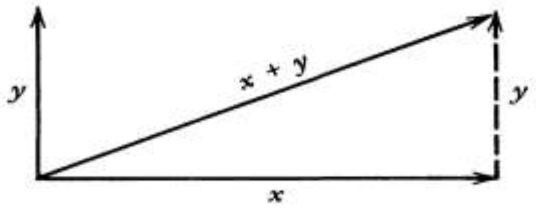

Fig. 24. Illustration of the Pythagorean theorem in the plane

3. If X in Prob. 2 is real, show that, conversely, the given relation implies that $x \perp y$. Show that this may not hold if X is complex. Give examples.

4. If an inner product space X is real, show that the condition $\|x\| = \|y\|$ implies $\langle x+y, x-y\rangle = 0$. What does this mean geometrically if $X = \mathbf{R}^2$? What does the condition imply if X is complex?

5. (Appolonius' identity) Verify by direct calculation that for any elements in an inner product space,

$$\|z-x\|^2 + \|z-y\|^2 = \tfrac{1}{2}\|x-y\|^2 + 2\,\|z - \tfrac{1}{2}(x+y)\|^2.$$

Show that this identity can also be obtained from the parallelogram equality.

6. Let $x \neq 0$ and $y \neq 0$. (a) If $x \perp y$, show that $\{x, y\}$ is a linearly independent set. (b) Extend the result to mutually orthogonal nonzero vectors $x_1, \cdots, x_m$.

7. If in an inner product space, $\langle x, u\rangle = \langle x, v\rangle$ for all x, show that $u = v$.

8. Prove (9).

9. Prove (10).

10. Let z_1 and z_2 denote complex numbers. Show that $\langle z_1, z_2\rangle = z_1\bar{z}_2$ defines an inner product, which yields the usual metric on the complex plane. Under what condition do we have orthogonality?

11. Let X be the vector space of all ordered pairs of complex numbers. Can we obtain the norm defined on X by

$$\|x\| = |\xi_1| + |\xi_2| \qquad [x = (\xi_1, \xi_2)]$$

from an inner product?

12. What is $\|x\|$ in 3.1-6 if $x = (\xi_1, \xi_2, \cdots)$, where (*a*) $\xi_n = 2^{-n/2}$, (*b*) $\xi_n = 1/n$?

13. Verify that for continuous functions the inner product in 3.1-5 satisfies (IP1) to (IP4).

14. Show that the norm on $C[a, b]$ is invariant under a linear transformation $t = \alpha\tau + \beta$. Use this to prove the statement in 3.1-8 by mapping $[a, b]$ onto $[0, 1]$ and then considering the functions defined by $\tilde{x}(\tau) = 1$, $\tilde{y}(\tau) = \tau$, where $\tau \in [0, 1]$.

15. If X is a finite dimensional vector space and (e_j) is a basis for X, show that an inner product on X is completely determined by its values $\gamma_{jk} = \langle e_j, e_k\rangle$. Can we choose such scalars γ_{jk} in a completely arbitrary fashion?

3.2 Further Properties of Inner Product Spaces

First of all, we should verify that (1) in the preceding section defines a norm:

(N1) and (N2) in Sec. 2.2 follow from (IP4). Furthermore, (N3) is obtained by the use of (IP2) and (IP3); in fact,

$$\|\alpha x\|^2 = \langle \alpha x, \alpha x\rangle = \alpha\bar{\alpha}\langle x, x\rangle = |\alpha|^2 \|x\|^2.$$

Finally, (N4) is included in

3.2-1 Lemma (Schwarz inequality, triangle inequality). *An inner product and the corresponding norm satisfy the Schwarz inequality and the triangle inequality as follows.*