If the Fourier series of $x$ converges for each $t$ and has the sum $x(t)$, then we write

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt). \tag{1}$$

Since $x$ is periodic of period $2\pi$, in (2) we may replace the interval of integration $[0, 2\pi]$ by any other interval of length $2\pi$, for instance $[-\pi, \pi]$.

Fourier series first arose in connection with physical problems considered by D. Bernoulli (vibrating string, 1753) and J. Fourier (heat conduction, 1822). These series help to represent complicated periodic phenomena in terms of simple periodic functions (cosine and sine). They have various physical applications in connection with differential equations (vibrations, heat conduction, potential problems, etc.).

From (2) we see that the determination of Fourier coefficients requires integration. To help those readers who have not seen Fourier series before, we consider as an illustration (see Fig. 34)

$$x(t) = \begin{cases} t & \text{if } -\pi/2 \leqq t < \pi/2 \\ \pi - t & \text{if } \pi/2 \leqq t < 3\pi/2 \end{cases}$$

and $x(t+2\pi) = x(t)$. From (2) we obtain $a_k = 0$ for $k = 0, 1, \cdots$ and, choosing $[-\pi/2, 3\pi/2]$ as a convenient interval of integration and integrating by parts,

$$\begin{aligned} b_k &= \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} t \sin kt \, dt + \frac{1}{\pi} \int_{\pi/2}^{3\pi/2} (\pi - t) \sin kt \, dt \\ &= -\frac{1}{\pi k} [t \cos kt] \Big|_{-\pi/2}^{\pi/2} + \frac{1}{\pi k} \int_{-\pi/2}^{\pi/2} \cos kt \, dt \\ &\quad - \frac{1}{\pi k} [(\pi - t) \cos kt] \Big|_{\pi/2}^{3\pi/2} - \frac{1}{\pi k} \int_{\pi/2}^{3\pi/2} \cos kt \, dt \\ &= \frac{4}{\pi k^2} \sin \frac{k\pi}{2}, \qquad k = 1, 2, \cdots. \end{aligned}$$

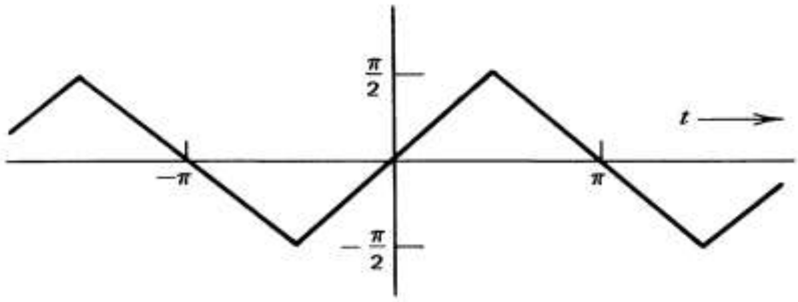


**Fig. 34.** Graph of the periodic function $x$, of period $2\pi$, given by $x(t)=t$ if $t \in [-\pi/2, \pi/2)$ and $x(t)=\pi-t$ if $t \in [\pi/2, 3\pi/2)$

Hence (1) takes the form

$$x(t)=\frac{4}{\pi}\left(\sin t-\frac{1}{3^2}\sin 3t+\frac{1}{5^2}\sin 5t-+\cdots\right).$$

The reader may graph the first three partial sums and compare them with the graph of $x$ in Fig. 34.

Returning to general Fourier series, we may ask how these series fit into our terminology and formalism introduced in the preceding section. Obviously, the cosine and sine functions in (1) are those of the sequences $(u_k)$ and $(v_k)$ in 3.4-5, that is

$$u_k(t)=\cos kt, \qquad v_k(t)=\sin kt.$$

Hence we may write (1) in the form

$$x(t)=a_0u_0(t)+\sum_{k=1}^{\infty}[a_ku_k(t)+b_kv_k(t)]. \tag{3}$$

We multiply (3) by a fixed $u_j$ and integrate over $t$ from 0 to $2\pi$. This means that we take the inner product by $u_j$ as defined in 3.4-5. We assume that termwise integration is permissible (uniform convergence would suffice) and use the orthogonality of $(u_k)$ and $(v_k)$ as well as the fact that $u_j \perp v_k$ for all $j$, $k$. Then we obtain

$$\begin{aligned}\langle x, u_j\rangle &= a_0\langle u_0, u_j\rangle+\sum[a_k\langle u_k, u_j\rangle+b_k\langle v_k, u_j\rangle]\\ &= a_j\langle u_j, u_j\rangle\\ &= a_j\,\|u_j\|^2=\begin{cases}2\pi a_0 & \text{if } j=0\\ \pi a_j & \text{if } j=1,2,\cdots,\end{cases}\end{aligned}$$

cf. (5), Sec. 3.4. Similarly, if we multiply (3) by $v_j$ and proceed as before, we arrive at

$$\langle x, v_j\rangle = b_j \|v_j\|^2 = \pi b_j$$

where $j = 1, 2, \cdots$. Solving for $a_j$ and $b_j$ and using the orthonormal sequences $(e_j)$ and $(\tilde{e}_j)$, where $e_j = \|u_j\|^{-1} u_j$ and $\tilde{e}_j = \|v_j\|^{-1} v_j$, we obtain

$$\begin{aligned} a_j &= \frac{1}{\|u_j\|^2} \langle x, u_j\rangle = \frac{1}{\|u_j\|} \langle x, e_j\rangle, \\ b_j &= \frac{1}{\|v_j\|^2} \langle x, v_j\rangle = \frac{1}{\|v_j\|} \langle x, \tilde{e}_j\rangle. \end{aligned} \tag{4}$$

This is identical with (2). It shows that in (3),

$$a_k u_k(t) = \frac{1}{\|u_k\|} \langle x, e_k\rangle u_k(t) = \langle x, e_k\rangle e_k(t)$$

and similarly for $b_k v_k(t)$. Hence we may write the Fourier series (1) in the form

$$x = \langle x, e_0\rangle e_0 + \sum_{k=1}^{\infty} [\langle x, e_k\rangle e_k + \langle x, \tilde{e}_k\rangle \tilde{e}_k]. \tag{5}$$

This justifies the term "Fourier coefficients" in the preceding section.

Concluding this example, we mention that the reader can find an introduction to Fourier series in W. Rogosinski (1959); cf. also R. V. Churchill (1963), pp. 77–112 and E. Kreyszig (1972), pp. 377–407. ∎

Our example concerns infinite series and raises the question how we can extend the consideration to other orthonormal sequences and what we can say about the convergence of corresponding series.

Given any orthonormal sequence $(e_k)$ in a Hilbert space $H$, we may consider series of the form

$$\sum_{k=1}^{\infty} \alpha_k e_k \tag{6}$$

where $\alpha_1, \alpha_2, \cdots$ are any scalars. As defined in Sec. 2.3, such a series *converges* and has the *sum* $s$ if there exists an $s \in H$ such that the sequence $(s_n)$ of the partial sums

$$s_n = \alpha_1 e_1 + \cdots + \alpha_n e_n$$

converges to $s$, that is, $\|s_n - s\| \longrightarrow 0$ as $n \longrightarrow \infty$.

**3.5-2 Theorem (Convergence).** *Let $(e_k)$ be an orthonormal sequence in a Hilbert space $H$. Then:*

**(a)** *The series (6) converges (in the norm on $H$) if and only if the following series converges:*

$$\sum_{k=1}^{\infty} |\alpha_k|^2. \tag{7}$$

**(b)** *If (6) converges, then the coefficients $\alpha_k$ are the Fourier coefficients $\langle x, e_k \rangle$, where $x$ denotes the sum of (6); hence in this case, (6) can be written*

$$x = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k. \tag{8}$$

**(c)** *For any $x \in H$, the series (6) with $\alpha_k = \langle x, e_k \rangle$ converges (in the norm of $H$).*

*Proof.* **(a)** Let

$$s_n = \alpha_1 e_1 + \cdots + \alpha_n e_n \qquad \text{and} \qquad \sigma_n = |\alpha_1|^2 + \cdots + |\alpha_n|^2.$$

Then, because of the orthonormality, for any $m$ and $n > m$,

$$\begin{aligned} \|s_n - s_m\|^2 &= \|\alpha_{m+1} e_{m+1} + \cdots + \alpha_n e_n\|^2 \\ &= |\alpha_{m+1}|^2 + \cdots + |\alpha_n|^2 = \sigma_n - \sigma_m. \end{aligned}$$

Hence $(s_n)$ is Cauchy in $H$ if and only if $(\sigma_n)$ is Cauchy in $\mathbf{R}$. Since $H$ and $\mathbf{R}$ are complete, the first statement of the theorem follows.

**(b)** Taking the inner product of $s_n$ and $e_j$ and using the orthonormality, we have

$$\langle s_n, e_j \rangle = \alpha_j \quad \text{for } j = 1, \cdots, k \qquad (k \leqq n \text{ and fixed}).$$

By assumption, $s_n \longrightarrow x$. Since the inner product is continuous (cf. Lemma 3.2-2),

$$\alpha_j = \langle s_n, e_j \rangle \quad \longrightarrow \quad \langle x, e_j \rangle \qquad (j \leqq k).$$

Here we can take $k$ $(\leqq n)$ as large as we please because $n \longrightarrow \infty$, so that we have $\alpha_j = \langle x, e_j \rangle$ for every $j = 1, 2, \cdots$.

**(c)** From the Bessel inequality in Theorem 3.4-6 we see that the series

$$\sum_{k=1}^{\infty} |\langle x, e_k \rangle|^2$$

converges. From this and $(a)$ we conclude that $(c)$ must hold. ∎

If an orthonormal family $(e_\kappa)$, $\kappa \in I$, in an inner product space $X$ is uncountable (since the index set $I$ is uncountable), we can still form the Fourier coefficients $\langle x, e_\kappa \rangle$ of an $x \in X$, where $\kappa \in I$. Now we use (12*), Sec. 3.4, to conclude that for each fixed $m = 1, 2, \cdots$ the number of Fourier coefficients such that $|\langle x, e_\kappa \rangle| > 1/m$ must be finite. This proves the remarkable

**3.5-3 Lemma (Fourier coefficients).** *Any $x$ in an inner product space $X$ can have at most countably many nonzero Fourier coefficients $\langle x, e_\kappa \rangle$ with respect to an orthonormal family $(e_\kappa)$, $\kappa \in I$, in $X$.*

Hence with any fixed $x \in H$ we can associate a series similar to (8),

$$\sum_{\kappa \in I} \langle x, e_\kappa \rangle e_\kappa \tag{9}$$

and we can arrange the $e_\kappa$ with $\langle x, e_\kappa \rangle \neq 0$ in a sequence $(e_1, e_2, \cdots)$, so that (9) takes the form (8). Convergence follows from Theorem 3.5-2. We show that the sum does not depend on the order in which those $e_\kappa$ are arranged in a sequence.

*Proof.* Let $(w_m)$ be a rearrangement of $(e_n)$. By definition this means that there is a bijective mapping $n \longmapsto m(n)$ of $\mathbf{N}$ onto itself such that corresponding terms of the two sequences are equal, that is,

$w_{m(n)} = e_n$. We set

$$\alpha_n = \langle x, e_n \rangle, \qquad \beta_m = \langle x, w_m \rangle$$

and

$$x_1 = \sum_{n=1}^{\infty} \alpha_n e_n, \qquad x_2 = \sum_{m=1}^{\infty} \beta_m w_m.$$

Then by Theorem 3.5-2($b$),

$$\alpha_n = \langle x, e_n \rangle = \langle x_1, e_n \rangle, \qquad \beta_m = \langle x, w_m \rangle = \langle x_2, w_m \rangle.$$

Since $e_n = w_{m(n)}$, we thus obtain

$$\begin{aligned} \langle x_1 - x_2, e_n \rangle &= \langle x_1, e_n \rangle - \langle x_2, w_{m(n)} \rangle \\ &= \langle x, e_n \rangle - \langle x, w_{m(n)} \rangle = 0 \end{aligned}$$

and similarly $\langle x_1 - x_2, w_m \rangle = 0$. This implies

$$\begin{aligned} \|x_1 - x_2\|^2 &= \langle x_1 - x_2, \sum \alpha_n e_n - \sum \beta_m w_m \rangle \\ &= \sum \bar{\alpha}_n \langle x_1 - x_2, e_n \rangle - \sum \bar{\beta}_m \langle x_1 - x_2, w_m \rangle = 0. \end{aligned}$$

Consequently, $x_1 - x_2 = 0$ and $x_1 = x_2$. Since the rearrangement $(w_m)$ of $(e_n)$ was arbitrary, this completes the proof. ∎

## Problems

**1.** If (6) converges with sum $x$, show that (7) has the sum $\|x\|^2$.

**2.** Derive from (1) and (2) a Fourier series representation of a function $\tilde{x}$ (function of $\tau$) of arbitrary period $p$.

**3.** Illustrate with an example that a convergent series $\sum \langle x, e_k \rangle e_k$ need not have the sum $x$.

**4.** If $(x_j)$ is a sequence in an inner product space $X$ such that the series $\|x_1\| + \|x_2\| + \cdots$ converges, show that $(s_n)$ is a Cauchy sequence, where $s_n = x_1 + \cdots + x_n$.

**5.** Show that in a Hilbert space $H$, convergence of $\sum \|x_j\|$ implies convergence of $\sum x_j$.

**6.** Let $(e_j)$ be an orthonormal sequence in a Hilbert space $H$. Show that if

$$x=\sum_{j=1}^{\infty} \alpha_j e_j, \qquad y=\sum_{j=1}^{\infty} \beta_j e_j, \qquad \text{then} \qquad \langle x, y\rangle=\sum_{j=1}^{\infty} \alpha_j \bar{\beta}_j,$$

the series being absolutely convergent.

**7.** Let $(e_k)$ be an orthonormal sequence in a Hilbert space $H$. Show that for every $x \in H$, the vector

$$y=\sum_{k=1}^{\infty} \langle x, e_k\rangle e_k$$

exists in $H$ and $x-y$ is orthogonal to every $e_k$.

**8.** Let $(e_k)$ be an orthonormal sequence in a Hilbert space $H$, and let $M=\operatorname{span}(e_k)$. Show that for any $x \in H$ we have $x \in \bar{M}$ if and only if $x$ can be represented by (6) with coefficients $\alpha_k=\langle x, e_k\rangle$.

**9.** Let $(e_n)$ and $(\tilde{e}_n)$ be orthonormal sequences in a Hilbert space $H$, and let $M_1=\operatorname{span}(e_n)$ and $M_2=\operatorname{span}(\tilde{e}_n)$. Using Prob. 8, show that $\bar{M}_1=\bar{M}_2$ if and only if

$$\text{(a)} \quad e_n=\sum_{m=1}^{\infty} \alpha_{nm} \tilde{e}_m, \qquad \text{(b)} \quad \tilde{e}_n=\sum_{m=1}^{\infty} \bar{\alpha}_{mn} e_m, \qquad \alpha_{nm}=\langle e_n, \tilde{e}_m\rangle.$$

**10.** Work out the details of the proof of Lemma 3.5-3.

## 3.6 Total Orthonormal Sets and Sequences

The truly interesting orthonormal sets in inner product spaces and Hilbert spaces are those which consist of "sufficiently many" elements so that every element in space can be represented or sufficiently accurately approximated by the use of those orthonormal sets. In finite dimensional ($n$-dimensional) spaces the situation is simple; all we need is an orthonormal set of $n$ elements. The question is what can be done to take care of infinite dimensional spaces, too. Relevant concepts are as follows.