3.6-1 Definition (Total orthonormal set). A *total set* (or *fundamental set*) in a normed space X is a subset $M \subset X$ whose span is dense in X (cf. 1.3-5). Accordingly, an orthonormal set (or sequence or family) in an inner product space X which is total in X is called a *total orthonormal set*[4] (or sequence or family, respectively) in X. ∎

M is total in X if and only if

$$\overline{\operatorname{span} M} = X.$$

This is obvious from the definition.

A total orthonormal family in X is sometimes called an *orthonormal basis* for X. However, it is important to note that this is not a basis, in the sense of algebra, for X as a vector space, unless X is finite dimensional.

In every Hilbert space $H \neq \{0\}$ there exists a total orthonormal set.

For a finite dimensional H this is clear. For an infinite dimensional separable H (cf. 1.3-5) it follows from the Gram-Schmidt process by (ordinary) induction. For a nonseparable H a (nonconstructive) proof results from Zorn's lemma, as we shall see in Sec. 4.1 where we introduce and explain the lemma for another purpose.

All total orthonormal sets in a given Hilbert space $H \neq \{0\}$ have the same cardinality. The latter is called the *Hilbert dimension* or *orthogonal dimension* of H. (If $H = \{0\}$, this dimension is defined to be 0.)

For a finite dimensional H the statement is clear since then the Hilbert dimension is the dimension in the sense of algebra. For an infinite dimensional separable H the statement will readily follow from Theorem 3.6-4 (below) and for a general H the proof would require somewhat more advanced tools from set theory; cf. E. Hewitt and K. Stromberg (1969), p. 246.

[4] Sometimes a *complete* orthonormal set, but we use "complete" only in the sense of Def. 1.4-3; this is preferable since we then avoid the use of the same word in connection with two entirely different concepts. [Moreover, some authors mean by "completeness" of an orthonormal set M the property expressed by (1) in Theorem 3.6-2. We do not adopt this terminology either.]

The following theorem shows that a total orthonormal set cannot be augmented to a more extensive orthonormal set by the adjunction of new elements.

3.6-2 Theorem (Totality). *Let M be a subset of an inner product space X. Then:*

(a) *If M is total in X, then there does not exist a nonzero $x \in X$ which is orthogonal to every element of M; briefly,*

$$x \perp M \quad \Longrightarrow \quad x = 0. \tag{1}$$

(b) *If X is complete, that condition is also sufficient for the totality of M in X.*

Proof. **(a)** Let H be the completion of X; cf. 3.2-3. Then X, regarded as a subspace of H, is dense in H. By assumption, M is total in X, so that span M is dense in X, hence dense in H. Lemma 3.3-7 now implies that the orthogonal complement of M in H is $\{0\}$. A fortiori, if $x \in X$ and $x \perp M$, then $x = 0$.

(b) If X is a Hilbert space and M satisfies that condition, so that $M^{\perp} = \{0\}$, then Lemma 3.3-7 implies that M is total in X. ∎

The completeness of X in (b) is essential. If X is not complete, there may not exist an orthonormal set $M \subset X$ such that M is total in X. An example was given by J. Dixmier (1953). Cf. also N. Bourbaki (1955), p. 155.

Another important criterion for totality can be obtained from the Bessel inequality (cf. 3.4-6). For this purpose we consider any given orthonormal set M in a Hilbert space H. From Lemma 3.5-3 we know that each fixed $x \in H$ has at most countably many nonzero Fourier coefficients, so that we can arrange these coefficients in a sequence, say, $\langle x, e_1 \rangle, \langle x, e_2 \rangle, \cdots$. The Bessel inequality is (cf. 3.4-6)

$$\sum_k |\langle x, e_k \rangle|^2 \leqq \|x\|^2 \qquad \textbf{(Bessel inequality)} \tag{2}$$

where the left-hand side is an infinite series or a finite sum. With the equality sign this becomes

$$\sum_k |\langle x, e_k\rangle|^2 = \|x\|^2 \qquad \text{(3)} \qquad \textbf{(Parseval relation)}$$

and yields another criterion for totality:

3.6-3 Theorem (Totality). *An orthonormal set M in a Hilbert space H is total in H if and only if for all $x \in H$ the Parseval relation* (3) *holds (summation over all nonzero Fourier coefficients of x with respect to M).*

Proof. (*a*) If M is not total, by Theorem 3.6-2 there is a nonzero $x \perp M$ in H. Since $x \perp M$, in (3) we have $\langle x, e_k\rangle = 0$ for all k, so that the left-hand side in (3) is zero, whereas $\|x\|^2 \neq 0$. This shows that (3) does not hold. Hence if (3) holds for all $x \in H$, then M must be total in H.

(*b*) Conversely, assume M to be total in H. Consider any $x \in H$ and its nonzero Fourier coefficients (cf. 3.5-3) arranged in a sequence $\langle x, e_1\rangle, \langle x, e_2\rangle, \cdots$, or written in some definite order if there are only finitely many of them. We now define y by

$$y = \sum_k \langle x, e_k\rangle e_k, \qquad \text{(4)}$$

noting that in the case of an infinite series, convergence follows from Theorem 3.5-2. Let us show that $x - y \perp M$. For every e_j occurring in (4) we have, using the orthonormality,

$$\langle x-y, e_j\rangle = \langle x, e_j\rangle - \sum_k \langle x, e_k\rangle\langle e_k, e_j\rangle = \langle x, e_j\rangle - \langle x, e_j\rangle = 0.$$

And for every $v \in M$ not contained in (4) we have $\langle x, v\rangle = 0$, so that

$$\langle x-y, v\rangle = \langle x, v\rangle - \sum_k \langle x, e_k\rangle\langle e_k, v\rangle = 0 - 0 = 0.$$

Hence $x - y \perp M$, that is, $x - y \in M^\perp$. Since M is total in H, we have $M^\perp = \{0\}$ from 3.3-7. Together, $x - y = 0$, that is, $x = y$. Using (4) and again the orthonormality, we thus obtain (3) from

$$\|x\|^2 = \left\langle \sum_k \langle x, e_k \rangle e_k, \sum_m \langle x, e_m \rangle e_m \right\rangle = \sum_k \langle x, e_k \rangle \overline{\langle x, e_k \rangle}.$$

This completes the proof. ∎

Let us turn to Hilbert spaces which are separable. By Def. 1.3-5 such a space has a countable subset which is dense in the space. Separable Hilbert spaces are simpler than nonseparable ones since they cannot contain uncountable orthonormal sets:

3.6-4 Theorem (Separable Hilbert spaces). *Let H be a Hilbert space. Then:*

(a) *If H is separable, every orthonormal set in H is countable.*

(b) *If H contains an orthonormal sequence which is total in H, then H is separable.*

Proof. **(a)** Let H be separable, B any dense set in H and M any orthonormal set. Then any two distinct elements x and y of M have distance $\sqrt{2}$ since

$$\|x-y\|^2 = \langle x-y, x-y \rangle = \langle x, x \rangle + \langle y, y \rangle = 2.$$

Hence spherical neighborhoods N_x of x and N_y of y of radius $\sqrt{2}/3$ are disjoint. Since B is dense in H, there is a $b \in B$ in N_x and a $\tilde{b} \in B$ in N_y and $b \neq \tilde{b}$ since $N_x \cap N_y = \varnothing$. Hence if M were uncountable, we would have uncountably many such pairwise disjoint spherical neighborhoods (for each $x \in M$ one of them), so that B would be uncountable. Since B was any dense set, this means that H would not contain a dense set which is countable, contradicting separability. From this we conclude that M must be countable.

(b) Let (e_k) be a total orthonormal sequence in H and A the set of all linear combinations

$$\gamma_1^{(n)} e_1 + \cdots + \gamma_n^{(n)} e_n \qquad n = 1, 2, \cdots$$

where $\gamma_k^{(n)} = a_k^{(n)} + i b_k^{(n)}$ and $a_k^{(n)}$ and $b_k^{(n)}$ are rational (and $b_k^{(n)} = 0$ if H is real). Clearly, A is countable. We prove that A is dense in H by showing that for every $x \in H$ and $\varepsilon > 0$ there is a $v \in A$ such that $\|x - v\| < \varepsilon$.

Since the sequence (e_k) is total in H, there is an n such that $Y_n = \text{span}\{e_1, \cdots, e_n\}$ contains a point whose distance from x is less than $\varepsilon/2$. In particular, $\|x - y\| < \varepsilon/2$ for the orthogonal projection y of x on Y_n, which is given by [cf. (8), Sec. 3.4]

$$y = \sum_{k=1}^{n} \langle x, e_k \rangle e_k.$$

Hence we have

$$\left\| x - \sum_{k=1}^{n} \langle x, e_k \rangle e_k \right\| < \frac{\varepsilon}{2}.$$

Since the rationals are dense on $\mathbf{R}$, for each $\langle x, e_k \rangle$ there is a $\gamma_k^{(n)}$ (with rational real and imaginary parts) such that

$$\left\| \sum_{k=1}^{n} [\langle x, e_k \rangle - \gamma_k^{(n)}] e_k \right\| < \frac{\varepsilon}{2}.$$

Hence $v \in A$ defined by

$$v = \sum_{k=1}^{n} \gamma_k^{(n)} e_k$$

satisfies

$$\begin{aligned} \|x - v\| &= \left\|x - \sum \gamma_k^{(n)} e_k\right\| \\ &\leqq \left\|x - \sum \langle x, e_k \rangle e_k\right\| + \left\|\sum \langle x, e_k \rangle e_k - \sum \gamma_k^{(n)} e_k\right\| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This proves that A is dense in H, and since A is countable, H is separable. ∎

For using Hilbert spaces in applications one must know what total orthonormal set or sets to choose in a specific situation and how to investigate properties of the elements of such sets. For certain function spaces this problem will be considered in the next section, which includes special functions of practical interest that arise in this context and have been investigated in very great detail. To conclude this section, let us point out that our present discussion has some further consequences which are of basic importance and can be formulated in terms of isomorphisms of Hilbert spaces. For this purpose we first remember from Sec. 3.2 the following.

An **isomorphism** of a Hilbert space H onto a Hilbert space $\tilde{H}$ over the same field is a bijective linear operator $T\colon H \longrightarrow \tilde{H}$ such that for all $x, y \in H$,

$$\langle Tx, Ty\rangle = \langle x, y\rangle. \tag{5}$$

H and $\tilde{H}$ are then called *isomorphic Hilbert spaces.* Since T is linear, it preserves the vector space structure, and (5) shows that T is isometric. From this and the bijectivity of T it follows that H and $\tilde{H}$ are algebraically as well as metrically indistinguishable; they are essentially the same, except for the nature of their elements, so that we may think of $\tilde{H}$ as being essentially H with a "tag" T attached to each vector x. Or we may regard H and $\tilde{H}$ as two copies (models) of the same abstract space, just as we often do in the case of n-dimensional Euclidean space.

Most exciting in this discussion is the fact that for each Hilbert dimension (cf. at the beginning of this section) there is just one abstract real Hilbert space and just one abstract complex Hilbert space. In other words, two abstract Hilbert spaces over the same field are distinguished only by their Hilbert dimension, a situation which generalizes that in the case of Euclidean spaces. This is the meaning of the following theorem.

3.6-5 Theorem (Isomorphism and Hilbert dimension). *Two Hilbert spaces H and $\tilde{H}$, both real or both complex, are isomorphic if and only if they have the same Hilbert dimension.*

Proof. (*a*) If H is isomorphic with $\tilde{H}$ and $T\colon H \longrightarrow \tilde{H}$ is an isomorphism, then (5) shows that orthonormal elements in H have orthonormal images under T. Since T is bijective, we thus conclude that T maps every total orthonormal set in H onto a total orthonormal set in $\tilde{H}$. Hence H and $\tilde{H}$ have the same Hilbert dimension.

(*b*) Conversely, suppose that H and $\tilde{H}$ have the same Hilbert dimension. The case $H=\{0\}$ and $\tilde{H}=\{0\}$ is trivial. Let $H\neq\{0\}$. Then $\tilde{H}\neq\{0\}$, and any total orthonormal sets M in H and $\tilde{M}$ in $\tilde{H}$ have the same cardinality, so that we can index them by the same index set $\{k\}$ and write $M=(e_k)$ and $\tilde{M}=(\tilde{e}_k)$.

To show that H and $\tilde{H}$ are isomorphic, we construct an isomorphism of H onto $\tilde{H}$. For every $x\in H$ we have

$$x=\sum_k \langle x, e_k\rangle e_k \tag{6}$$

where the right-hand side is a finite sum or an infinite series (cf. 3.5-3), and $\sum_k |\langle x, e_k\rangle|^2<\infty$ by the Bessel inequality. Defining

$$\tilde{x}=Tx=\sum_k \langle x, e_k\rangle \tilde{e}_k \tag{7}$$

we thus have convergence by 3.5-2, so that $\tilde{x}\in\tilde{H}$. The operator T is linear since the inner product is linear with respect to the first factor. T is isometric, because by first using (7) and then (6) we obtain

$$\|\tilde{x}\|^2=\|Tx\|^2=\sum_k |\langle x, e_k\rangle|^2=\|x\|^2.$$

From this and (9), (10) in Sec. 3.1 we see that T preserves the inner product. Furthermore, isometry implies injectivity. In fact, if $Tx=Ty$, then

$$\|x-y\|=\|T(x-y)\|=\|Tx-Ty\|=0,$$

so that $x=y$ and T is injective by 2.6-10.

We finally show that T is surjective. Given any

$$\tilde{x}=\sum_k \alpha_k\tilde{e}_k$$

in $\tilde{H}$, we have $\sum|\alpha_k|^2<\infty$ by the Bessel inequality. Hence

$$\sum_k \alpha_k e_k$$

is a finite sum or a series which converges to an $x\in H$ by 3.5-2, and $\alpha_k=\langle x, e_k\rangle$ by the same theorem. We thus have $\tilde{x}=Tx$ by (7). Since $\tilde{x}\in\tilde{H}$ was arbitrary, this shows that T is surjective. ∎

Problems

1. If F is an orthonormal basis in an inner product space X, can we represent every $x \in X$ as a linear combination of elements of F? (By definition, a linear combination consists of finitely many terms.)

2. Show that if the orthogonal dimension of a Hilbert space H is finite, it equals the dimension of H regarded as a vector space; conversely, if the latter is finite, show that so is the former.

3. From what theorem of elementary geometry does (3) follow in the case of Euclidean n-space?

4. Derive from (3) the following formula (which is often called the *Parseval relation*).

$$\langle x, y\rangle = \sum_k \langle x, e_k\rangle \overline{\langle y, e_k\rangle}.$$

5. Show that an orthonormal family (e_κ), $\kappa \in I$, in a Hilbert space H is total if and only if the relation in Prob. 4 holds for every x and y in H.

6. Let H be a separable Hilbert space and M a countable dense subset of H. Show that H contains a total orthonormal sequence which can be obtained from M by the Gram-Schmidt process.

7. Show that if a Hilbert space H is separable, the existence of a total orthonormal set in H can be proved without the use of Zorn's lemma.

8. Show that for any orthonormal sequence F in a separable Hilbert space H there is a total orthonormal sequence $\tilde{F}$ which contains F.

9. Let M be a total set in an inner product space X. If $\langle v, x\rangle = \langle w, x\rangle$ for all $x \in M$, show that $v = w$.

10. Let M be a subset of a Hilbert space H, and let $v, w \in H$. Suppose that $\langle v, x\rangle = \langle w, x\rangle$ for all $x \in M$ implies $v = w$. If this holds for all $v, w \in H$, show that M is total in H.

3.7 Legendre, Hermite and Laguerre Polynomials

The theory of Hilbert spaces has applications to various solid topics in analysis. In the present section we discuss some total orthogonal and orthonormal sequences which are used quite frequently in connection