

function in (8) by v , for simplicity, and integrate m times by parts. Then, by (7b),

$$\begin{aligned}
 (-1)^n \int_{-\infty}^{+\infty} e^{-t^2} H_m(t) H_n(t) dt &= \int_{-\infty}^{+\infty} H_m(t) v^{(n)} dt \\
 &= H_m(t) v^{(n-1)} \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} 2m H_{m-1}(t) v^{(n-1)} dt \\
 &= -2m \int_{-\infty}^{+\infty} H_{m-1}(t) v^{(n-1)} dt \\
 &= \dots \\
 &= (-1)^m 2^m m! \int_{-\infty}^{+\infty} H_0(t) v^{(n-m)} dt.
 \end{aligned}$$

Here $H_0(t) = 1$. If $m < n$, integrating once more, we obtain 0 since v and its derivatives approach zero as $t \rightarrow +\infty$ or $t \rightarrow -\infty$. This proves orthogonality of (e_n) . We prove (8) for $m = n$, which entails $\|e_n\| = 1$ by (7a). If $m = n$, for the last integral, call it J , we obtain

$$J = \int_{-\infty}^{+\infty} e^{-t^2} dt = \sqrt{\pi}.$$

This is a familiar result. To verify it, consider J^2 , use polar coordinates r, θ and $ds dt = r dr d\theta$, finding

$$\begin{aligned}
 J^2 &= \int_{-\infty}^{+\infty} e^{-s^2} ds \int_{-\infty}^{+\infty} e^{-t^2} dt = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{-(s^2+t^2)} ds dt \\
 &= \int_0^{2\pi} \int_0^{+\infty} e^{-r^2} r dr d\theta \\
 &= 2\pi \cdot \frac{1}{2} = \pi.
 \end{aligned}$$

This proves (8), hence the orthonormality of (e_n) . ■

Classically speaking, one often expresses (8) by saying that the H_n 's form an orthogonal sequence with respect to the *weight function* w^2 , where w is the function defined at the beginning.

It can be shown that (e_n) defined by (7a), (7b) is total in the real space $L^2(-\infty, +\infty)$. Hence this space is separable. (Cf. 3.6-4.)

We finally mention that the Hermite polynomials H_n satisfy the Hermite differential equation

$$(9) \quad H_n'' - 2tH_n' + 2nH_n = 0.$$

Warning. Unfortunately, the terminology in the literature is not unique. In fact, the functions He_n defined by

$$He_0(t) = 1, \quad He_n(t) = (-1)^n e^{t^2/2} \frac{d^n}{dt^n} (e^{-t^2/2}) \quad n = 1, 2, \dots$$

are also called “Hermite polynomials” and, to make things worse, are sometimes denoted by H_n .

An application of Hermite polynomials in quantum mechanics will be considered in Sec. 11.3.

3.7-3 Laguerre polynomials. A total orthonormal sequence in $L^2(-\infty, b]$ or $L^2[a, +\infty)$ can be obtained from such a sequence in $L^2[0, +\infty)$ by the transformations $t = b - s$ and $t = s + a$, respectively.

We consider $L^2[0, +\infty)$. Applying the Gram-Schmidt process to the sequence defined by

$$e^{-t/2}, \quad te^{-t/2}, \quad t^2e^{-t/2}, \quad \dots$$

we obtain an orthonormal sequence (e_n) . It can be shown that (e_n) is total in $L^2[0, +\infty)$ and is given by (Fig. 37)

$$(10a) \quad e_n(t) = e^{-t/2} L_n(t) \quad n = 0, 1, \dots$$

where the *Laguerre polynomial of order n* is defined by

$$(10b) \quad L_0(t) = 1, \quad L_n(t) = \frac{e^t}{n!} \frac{d^n}{dt^n} (t^n e^{-t}) \quad n = 1, 2, \dots,$$

that is,

$$(10c) \quad L_n(t) = \sum_{j=0}^n \frac{(-1)^j}{j!} \binom{n}{j} t^j. \quad \text{>}$$

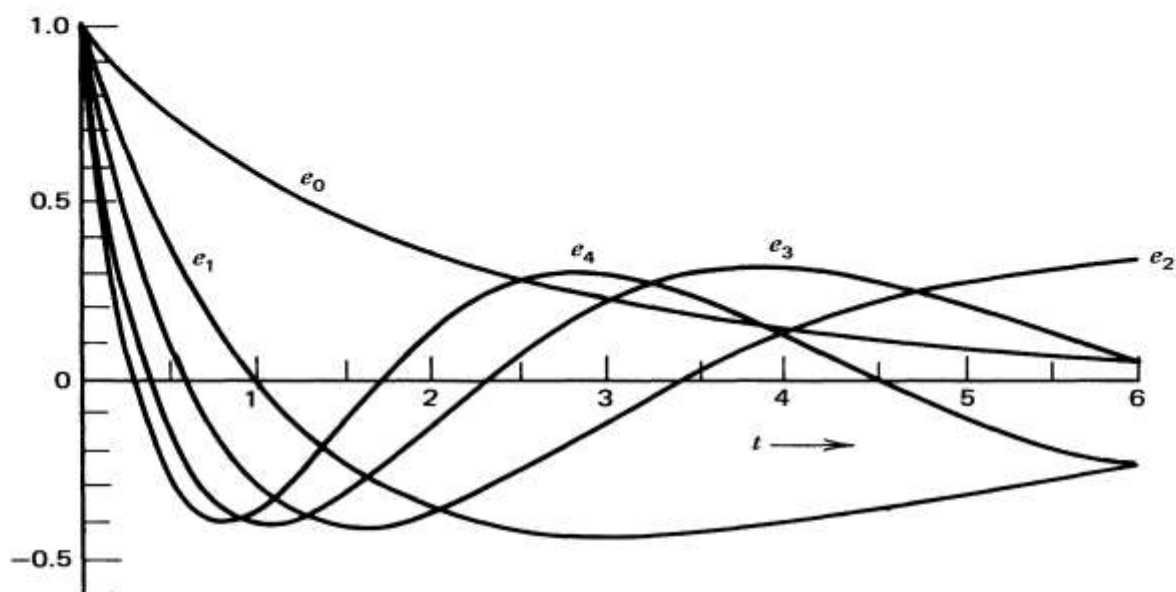


Fig. 37. Functions e_n in (10a) involving Laguerre polynomials

Explicit expressions for the first few Laguerre polynomials are

$$\begin{aligned}
 L_0(t) &= 1 & L_1(t) &= 1 - t \\
 (10^*) \quad L_2(t) &= 1 - 2t + \frac{1}{2}t^2 & L_3(t) &= 1 - 3t + \frac{3}{2}t^2 - \frac{1}{6}t^3 \\
 L_4(t) &= 1 - 4t + 3t^2 - \frac{2}{3}t^3 + \frac{1}{24}t^4.
 \end{aligned}$$

The Laguerre polynomials L_n are solutions of the *Laguerre differential equation*

$$(11) \quad tL_n'' + (1-t)L_n' + nL_n = 0.$$

For further details, see A. Erdélyi et al. (1953–55); cf. also R. Courant and D. Hilbert (1953–62), vol. I.

Problems

1. Show that the Legendre differential equation can be written

$$[(1-t^2)P_n']' = -n(n+1)P_n.$$

Multiply this by P_m . Multiply the corresponding equation for P_m by

$-P_n$ and add the two equations. Integrating the resulting equation from -1 to 1 , show that (P_n) is an orthogonal sequence in the space $L^2[-1, 1]$.

2. Derive (2c) from (2b).

3. **(Generating function)** Show that

$$\frac{1}{\sqrt{1-2tw+w^2}} = \sum_{n=0}^{\infty} P_n(t)w^n.$$

The function on the left is called a *generating function* of the Legendre polynomials. Generating functions are useful in connection with various special functions; cf. R. Courant and D. Hilbert (1953–62), A. Erdélyi et al. (1953–55).

4. Show that

$$\frac{1}{r} = \frac{1}{\sqrt{r_1^2 + r_2^2 - 2r_1r_2 \cos \theta}} = \frac{1}{r_2} \sum_{n=0}^{\infty} P_n(\cos \theta) \left(\frac{r_1}{r_2}\right)^n$$

where r is the distance between given points A_1 and A_2 in \mathbf{R}^3 , as shown in Fig. 38, and $r_2 > 0$. (This formula is useful in potential theory.)

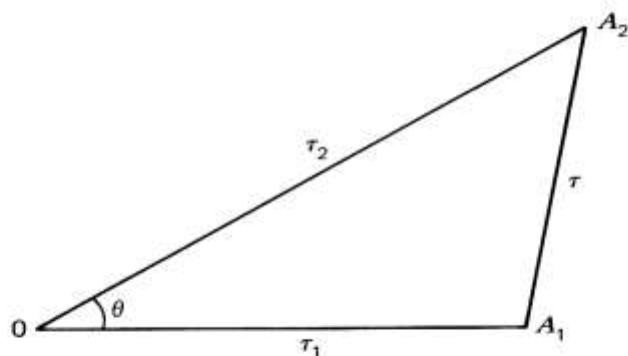


Fig. 38. Problem 4

5. Obtain the Legendre polynomials by the power series method as follows. Substitute $x(t) = c_0 + c_1t + c_2t^2 + \dots$ into Legendre's equation and show that by determining the coefficients one obtains the solution $x = c_0x_1 + c_1x_2$, where

$$x_1(t) = 1 - \frac{n(n+1)}{2!}t^2 + \frac{(n-2)n(n+1)(n+3)}{4!}t^4 - + \dots$$

and

$$x_2 = t - \frac{(n-1)(n+2)}{3!} t^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} t^5 - \dots$$

Show that for $n \in \mathbf{N}$, one of these two functions reduces to a polynomial, which agrees with P_n if one chooses $c_n = (2n)!/2^n (n!)^2$ as the coefficient of t^n .

6. (Generating function) Show that

$$\exp(2wt - w^2) = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(t) w^n.$$

The function on the left is called a *generating function* of the Hermite polynomials.

7. Using (7b), show that

$$H_{n+1}(t) = 2tH_n(t) - H_n'(t).$$

8. Differentiating the generating function in Prob. 6 with respect to t , show that

$$H_n'(t) = 2nH_{n-1}(t) \quad (n \geq 1)$$

and, using Prob. 7, show that H_n satisfies the Hermite differential equation.

9. Solve the differential equation $y'' + (2n+1-t^2)y = 0$ in terms of Hermite polynomials.

10. Using Prob. 8, show that

$$(e^{-t^2} H_n')' = -2ne^{-t^2} H_n.$$

Using this and the method explained in Prob. 1, show that the functions defined by (7a) are orthogonal on \mathbf{R} .

11. (Generating function) Using (10c), show that

$$\psi(t, w) = \frac{1}{1-w} \exp\left[-\frac{tw}{1-w}\right] = \sum_{n=0}^{\infty} L_n(t) w^n.$$

12. Differentiating ψ in Prob. 11 with respect to w , show that

$$(a) \quad (n+1)L_{n+1}(t) - (2n+1-t)L_n(t) + nL_{n-1}(t) = 0.$$

Differentiating ψ with respect to t , show that

$$(b) \quad L_{n-1}(t) = L'_{n-1}(t) - L'_n(t).$$

13. Using Prob. 12, show that

$$(c) \quad tL'_n(t) = nL_n(t) - nL_{n-1}(t).$$

Using this and (b) in Prob. 12, show that L_n satisfies Laguerre's differential equation (11).

14. Show that the functions in (10a) have norm 1.

15. Show that the functions in (10a) constitute an orthogonal sequence in the space $L^2[0, +\infty)$.

3.8 Representation of Functionals on Hilbert Spaces

It is of practical importance to know the general form of bounded linear functionals on various spaces. This was pointed out and explained in Sec. 2.10. For general Banach spaces such formulas and their derivation can sometimes be complicated. However, for a Hilbert space the situation is surprisingly simple:

3.8-1 Riesz's Theorem (Functionals on Hilbert spaces). *Every bounded linear functional f on a Hilbert space H can be represented in terms of the inner product, namely,*

$$(1) \quad f(x) = \langle x, z \rangle$$

where z depends on f , is uniquely determined by f and has norm

$$(2) \quad \|z\| = \|f\|.$$

Proof. We prove that

- (a) f has a representation (1),
- (b) z in (1) is unique,
- (c) formula (2) holds.

The details are as follows.

(a) If $f = 0$, then (1) and (2) hold if we take $z = 0$. Let $f \neq 0$. To motivate the idea of the proof, let us ask what properties z must have if a representation (1) exists. First of all, $z \neq 0$ since otherwise $f = 0$. Second, $\langle x, z \rangle = 0$ for all x for which $f(x) = 0$, that is, for all x in the null space $\mathcal{N}(f)$ of f . Hence $z \perp \mathcal{N}(f)$. This suggests that we consider $\mathcal{N}(f)$ and its orthogonal complement $\mathcal{N}(f)^\perp$.

$\mathcal{N}(f)$ is a vector space by 2.6-9 and is closed by 2.7-10. Furthermore, $f \neq 0$ implies $\mathcal{N}(f) \neq H$, so that $\mathcal{N}(f)^\perp \neq \{0\}$ by the projection theorem 3.3-4. Hence $\mathcal{N}(f)^\perp$ contains a $z_0 \neq 0$. We set

$$v = f(x)z_0 - f(z_0)x$$

where $x \in H$ is arbitrary. Applying f , we obtain

$$f(v) = f(x)f(z_0) - f(z_0)f(x) = 0.$$

This shows that $v \in \mathcal{N}(f)$. Since $z_0 \perp \mathcal{N}(f)$, we have

$$\begin{aligned} 0 &= \langle v, z_0 \rangle = \langle f(x)z_0 - f(z_0)x, z_0 \rangle \\ &= f(x)\langle z_0, z_0 \rangle - f(z_0)\langle x, z_0 \rangle. \end{aligned}$$

Noting that $\langle z_0, z_0 \rangle = \|z_0\|^2 \neq 0$, we can solve for $f(x)$. The result is

$$f(x) = \frac{f(z_0)}{\langle z_0, z_0 \rangle} \langle x, z_0 \rangle.$$

This can be written in the form (1), where

$$z = \frac{\overline{f(z_0)}}{\langle z_0, z_0 \rangle} z_0.$$

Since $x \in H$ was arbitrary, (1) is proved.

(b) We prove that z in (1) is unique. Suppose that for all $x \in H$,

$$f(x) = \langle x, z_1 \rangle = \langle x, z_2 \rangle.$$

Then $\langle x, z_1 - z_2 \rangle = 0$ for all x . Choosing the particular $x = z_1 - z_2$, we have

$$\langle x, z_1 - z_2 \rangle = \langle z_1 - z_2, z_1 - z_2 \rangle = \|z_1 - z_2\|^2 = 0.$$

Hence $z_1 - z_2 = 0$, so that $z_1 = z_2$, the uniqueness.

(c) We finally prove (2). If $f = 0$, then $z = 0$ and (2) holds. Let $f \neq 0$. Then $z \neq 0$. From (1) with $x = z$ and (3) in Sec. 2.8 we obtain

$$\|z\|^2 = \langle z, z \rangle = f(z) \leq \|f\| \|z\|.$$

Division by $\|z\| \neq 0$ yields $\|z\| \leq \|f\|$. It remains to show that $\|f\| \leq \|z\|$. From (1) and the Schwarz inequality (Sec. 3.2) we see that

$$|f(x)| = |\langle x, z \rangle| \leq \|x\| \|z\|.$$

This implies

$$\|f\| = \sup_{\|x\|=1} |\langle x, z \rangle| \leq \|z\|. \quad \blacksquare$$

The idea of the uniqueness proof in part (b) is worth noting for later use:

3.8-2 Lemma (Equality). *If $\langle v_1, w \rangle = \langle v_2, w \rangle$ for all w in an inner product space X , then $v_1 = v_2$. In particular, $\langle v_1, w \rangle = 0$ for all $w \in X$ implies $v_1 = 0$.*

Proof. By assumption, for all w ,

$$\langle v_1 - v_2, w \rangle = \langle v_1, w \rangle - \langle v_2, w \rangle = 0.$$

For $w = v_1 - v_2$ this gives $\|v_1 - v_2\|^2 = 0$. Hence $v_1 - v_2 = 0$, so that $v_1 = v_2$. In particular, $\langle v_1, w \rangle = 0$ with $w = v_1$ gives $\|v_1\|^2 = 0$, so that $v_1 = 0$. \blacksquare