The practical usefulness of bounded linear functionals on Hilbert spaces results to a large extent from the simplicity of the Riesz representation (1).

Furthermore, (1) is quite important in the theory of operators on Hilbert spaces. In particular, this refers to the Hilbert-adjoint operator T^* of a bounded linear operator T which we shall define in the next section. For this purpose we need a preparation which is of general interest, too. We begin with the following definition.

3.8-3 Definition (Sesquilinear form). Let X and Y be vector spaces over the same field K ($=\mathbf{R}$ or $\mathbf{C}$). Then a *sesquilinear form* (or *sesquilinear functional*) h on $X \times Y$ is a mapping

$$h\colon X \times Y \longrightarrow K$$

such that for all $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$ and all scalars α, β,

$$(3)\quad \begin{aligned} &\text{(a)} && h(x_1+x_2, y) = h(x_1, y) + h(x_2, y) \\ &\text{(b)} && h(x, y_1+y_2) = h(x, y_1) + h(x, y_2) \\ &\text{(c)} && h(\alpha x, y) = \alpha h(x, y) \\ &\text{(d)} && h(x, \beta y) = \bar{\beta} h(x, y). \end{aligned}$$

Hence h is *linear* in the first argument and *conjugate linear* in the second one. If X and Y are real ($K = \mathbf{R}$), then (3d) is simply

$$h(x, \beta y) = \beta h(x, y)$$

and h is called *bilinear* since it is linear in both arguments.

If X and Y are normed spaces and if there is a real number c such that for all x, y

$$(4)\qquad |h(x, y)| \leqq c\, \|x\|\, \|y\|,$$

then h is said to be *bounded*, and the number

$$\textbf{(5)}\qquad \|h\| = \sup_{\substack{x \in X-\{0\} \\ y \in Y-\{0\}}} \frac{|h(x, y)|}{\|x\|\, \|y\|} = \sup_{\substack{\|x\|=1 \\ \|y\|=1}} |h(x, y)|$$

is called the *norm* of h. ∎

For example, the inner product is sesquilinear and bounded. Note that from (4) and (5) we have

$$|h(x, y)| \leqq \|h\| \, \|x\| \, \|y\|. \tag{6}$$

The term "sesquilinear" was motivated in Sec. 3.1. In Def. 3.8-3, both words "form" and "functional" are common, the usage of one or the other being largely a matter of individual taste. Perhaps it is slightly preferable to use "form" in this two-variable case and reserve the word "functional" to one-variable cases such as that in Theorem 3.8-1. This is what we shall do.

It is quite interesting that from Theorem 3.8-1 we can obtain a general representation of sesquilinear forms on Hilbert spaces as follows.

3.8-4 Theorem (Riesz representation). *Let H_1, H_2 be Hilbert spaces and*

$$h\colon H_1 \times H_2 \longrightarrow K$$

a bounded sesquilinear form. Then h has a representation

$$h(x, y) = \langle Sx, y \rangle \tag{7}$$

where $S\colon H_1 \longrightarrow H_2$ is a bounded linear operator. S is uniquely determined by h and has norm

$$\|S\| = \|h\|. \tag{8}$$

Proof. We consider $\overline{h(x, y)}$. This is linear in y, because of the bar. To make Theorem 3.8-1 applicable, we keep x fixed. Then that theorem yields a representation in which y is variable, say,

$$\overline{h(x, y)} = \langle y, z \rangle.$$

Hence

$$h(x, y) = \langle z, y \rangle. \tag{9}$$

Here $z \in H_2$ is unique but, of course, depends on our fixed $x \in H_1$. It

follows that (9) with variable x defines an operator

$$S\colon\ H_1 \longrightarrow H_2 \qquad \text{given by} \qquad z = Sx.$$

Substituting $z = Sx$ in (9), we have (7).

S is linear. In fact, its domain is the vector space H_1, and from (7) and the sesquilinearity we obtain

$$\begin{aligned}\langle S(\alpha x_1 + \beta x_2), y\rangle &= h(\alpha x_1 + \beta x_2, y)\\ &= \alpha h(x_1, y) + \beta h(x_2, y)\\ &= \alpha\langle Sx_1, y\rangle + \beta\langle Sx_2, y\rangle\\ &= \langle \alpha Sx_1 + \beta Sx_2, y\rangle\end{aligned}$$

for all y in H_2, so that by Lemma 3.8-2,

$$S(\alpha x_1 + \beta x_2) = \alpha Sx_1 + \beta Sx_2.$$

S is bounded. Indeed, leaving aside the trivial case $S = 0$, we have from (5) and (7)

$$\|h\| = \sup_{\substack{x\neq 0\\ y\neq 0}} \frac{|\langle Sx, y\rangle|}{\|x\|\,\|y\|} \geqq \sup_{\substack{x\neq 0\\ Sx\neq 0}} \frac{|\langle Sx, Sx\rangle|}{\|x\|\,\|Sx\|} = \sup_{x\neq 0} \frac{\|Sx\|}{\|x\|} = \|S\|.$$

This proves boundedness. Moreover, $\|h\| \geqq \|S\|$.

We now obtain (8) by noting that $\|h\| \leqq \|S\|$ follows by an application of the Schwarz inequality:

$$\|h\| = \sup_{\substack{x\neq 0\\ y\neq 0}} \frac{|\langle Sx, y\rangle|}{\|x\|\,\|y\|} \leqq \sup_{x\neq 0} \frac{\|Sx\|\,\|y\|}{\|x\|\,\|y\|} = \|S\|.$$

S is unique. In fact, assuming that there is a linear operator $T\colon\ H_1 \longrightarrow H_2$ such that for all $x \in H_1$ and $y \in H_2$ we have

$$h(x, y) = \langle Sx, y\rangle = \langle Tx, y\rangle,$$

we see that $Sx = Tx$ by Lemma 3.8-2 for all $x \in H_1$. Hence $S = T$ by definition. ∎

Problems

1. **(Space $\mathbf{R}^3$)** Show that any linear functional f on $\mathbf{R}^3$ can be represented by a dot product:

$$f(x) = x \cdot z = \xi_1\zeta_1 + \xi_2\zeta_2 + \xi_3\zeta_3.$$

2. **(Space l^2)** Show that every bounded linear functional f on l^2 can be represented in the form

$$f(x) = \sum_{j=1}^{\infty} \xi_j\bar{\zeta}_j \qquad [z = (\zeta_j) \in l^2].$$

3. If z is any fixed element of an inner product space X, show that $f(x) = \langle x, z\rangle$ defines a bounded linear functional f on X, of norm $\|z\|$.

4. Consider Prob. 3. If the mapping $X \longrightarrow X'$ given by $z \longmapsto f$ is surjective, show that X must be a Hilbert space.

5. Show that the dual space of the real space l^2 is l^2. (Use 3.8-1.)

6. Show that Theorem 3.8-1 defines an isometric bijection $T\colon H \longrightarrow H'$, $z \longmapsto f_z = \langle \cdot, z\rangle$ which is not linear but *conjugate linear*, that is, $\alpha z + \beta v \longmapsto \bar{\alpha} f_z + \bar{\beta} f_v$.

7. Show that the dual space H' of a Hilbert space H is a Hilbert space with inner product $\langle \cdot, \cdot\rangle_1$ defined by

$$\langle f_z, f_v\rangle_1 = \overline{\langle z, v\rangle} = \langle v, z\rangle,$$

where $f_z(x) = \langle x, z\rangle$, etc.

8. Show that any Hilbert space H is isomorphic (cf. Sec. 3.6) with its second dual space $H'' = (H')'$. (This property is called *reflexivity* of H. It will be considered in more detail for normed spaces in Sec. 4.6.)

9. **(Annihilator)** Explain the relation between M^a in Prob. 13, Sec. 2.10, and $M^\perp$ in Sec. 3.3 in the case of a subset $M \neq \varnothing$ of a Hilbert space H.

10. Show that the inner product $\langle \cdot, \cdot\rangle$ on an inner product space X is a bounded sesquilinear form h. What is $\|h\|$ in this case?

11. If X is a vector space and h a sesquilinear form on $X \times X$, show that $f_1(x) = h(x, y_0)$ with fixed y_0 defines a linear functional f_1 on X, and so does $f_2(y) = \overline{h(x_0, y)}$ with fixed x_0.

12. Let X and Y be normed spaces. Show that a bounded sesquilinear form h on $X \times Y$ is jointly continuous in both variables.

13. **(Hermitian form)** Let X be a vector space over a field K. A *Hermitian sesquilinear form* or, simply, *Hermitian form* h on $X \times X$ is a mapping $h\colon X \times X \longrightarrow K$ such that for all $x, y, z \in X$ and $\alpha \in K$,

$$h(x+y, z) = h(x, z) + h(y, z)$$

$$h(\alpha x, y) = \alpha h(x, y)$$

$$h(x, y) = \overline{h(y, x)}.$$

What is the last condition if $K = \mathbf{R}$? What condition must be added for h to be an inner product on X?

14. **(Schwarz inequality)** Let X be a vector space and h a Hermitian form on $X \times X$. This form is said to be *positive semidefinite* if $h(x, x) \geqq 0$ for all $x \in X$. Show that then h satisfies the *Schwarz inequality*

$$|h(x, y)|^2 \leqq h(x, x)h(y, y).$$

15. **(Seminorm)** If h satisfies the conditions in Prob. 14, show that

$$p(x) = \sqrt{h(x, x)} \qquad (\geqq 0)$$

defines a seminorm on X. (Cf. Prob. 12, Sec. 2.3.)

3.9 Hilbert-Adjoint Operator

The results of the previous section will now enable us to introduce the Hilbert-adjoint operator of a bounded linear operator on a Hilbert space. This operator was suggested by problems in matrices and linear differential and integral equations. We shall see that it also helps to define three important classes of operators (called *self-adjoint*, *unitary*

and *normal operators*) which have been studied extensively because they play a key role in various applications.

3.9-1 Definition (Hilbert-adjoint operator T^*). Let $T\colon H_1 \longrightarrow H_2$ be a bounded linear operator, where H_1 and H_2 are Hilbert spaces. Then the *Hilbert-adjoint operator* T^* of T is the operator

$$T^*\colon H_2 \longrightarrow H_1$$

such that[5] for all $x \in H_1$ and $y \in H_2$,

$$\langle Tx, y\rangle = \langle x, T^* y\rangle. \tag{1}$$

∎

Of course, we should first show that this definition makes sense, that is, we should prove that for a given T such a T^* does exist:

3.9-2 Theorem (Existence). *The Hilbert-adjoint operator T^* of T in Def.* 3.9-1 *exists, is unique and is a bounded linear operator with norm*

$$\|T^*\| = \|T\|. \tag{2}$$

Proof. The formula

$$h(y, x) = \langle y, Tx\rangle \tag{3}$$

defines a sesquilinear form on $H_2 \times H_1$ because the inner product is sesquilinear and T is linear. In fact, conjugate linearity of the form is seen from

$$\begin{aligned} h(y, \alpha x_1 + \beta x_2) &= \langle y, T(\alpha x_1 + \beta x_2)\rangle \\ &= \langle y, \alpha Tx_1 + \beta Tx_2\rangle \\ &= \bar{\alpha}\langle y, Tx_1\rangle + \bar{\beta}\langle y, Tx_2\rangle \\ &= \bar{\alpha} h(y, x_1) + \bar{\beta} h(y, x_2). \end{aligned}$$

h is bounded. Indeed, by the Schwarz inequality,

$$|h(y, x)| = |\langle y, Tx\rangle| \leqq \|y\|\,\|Tx\| \leqq \|T\|\,\|x\|\,\|y\|.$$

[5] We may denote inner products on H_1 and H_2 by the same symbol since the factors show to which space an inner product refers.

This also implies $\|h\| \leqq \|T\|$. Moreover we have $\|h\| \geqq \|T\|$ from

$$\|h\| = \sup_{\substack{x \neq 0 \\ y \neq 0}} \frac{|\langle y, Tx \rangle|}{\|y\| \, \|x\|} \geqq \sup_{\substack{x \neq 0 \\ Tx \neq 0}} \frac{|\langle Tx, Tx \rangle|}{\|Tx\| \, \|x\|} = \|T\|.$$

Together,

$$\|h\| = \|T\|. \tag{4}$$

Theorem 3.8-4 gives a Riesz representation for h; writing T^* for S, we have

$$h(y, x) = \langle T^* y, x \rangle, \tag{5}$$

and we know from that theorem that T^*: $H_2 \longrightarrow H_1$ is a uniquely determined bounded linear operator with norm [cf. (4)]

$$\|T^*\| = \|h\| = \|T\|.$$

This proves (2). Also $\langle y, Tx \rangle = \langle T^* y, x \rangle$ by comparing (3) and (5), so that we have (1) by taking conjugates, and we now see that T^* is in fact the operator we are looking for. ∎

In our study of properties of Hilbert-adjoint operators it will be convenient to make use of the following lemma.

3.9-3 Lemma (Zero operator). *Let X and Y be inner product spaces and Q: $X \longrightarrow Y$ a bounded linear operator. Then:*

(a) *$Q = 0$ if and only if $\langle Qx, y \rangle = 0$ for all $x \in X$ and $y \in Y$.*

(b) *If Q: $X \longrightarrow X$, where X is complex, and $\langle Qx, x \rangle = 0$ for all $x \in X$, then $Q = 0$.*

Proof. **(a)** $Q = 0$ means $Qx = 0$ for all x and implies

$$\langle Qx, y \rangle = \langle 0, y \rangle = 0 \langle w, y \rangle = 0.$$

Conversely, $\langle Qx, y \rangle = 0$ for all x and y implies $Qx = 0$ for all x by 3.8-2, so that $Q = 0$ by definition.

(b) By assumption, $\langle Qv, v \rangle = 0$ for every $v = \alpha x + y \in X$,