that is,

$$\begin{aligned} 0 &= \langle Q(\alpha x + y), \alpha x + y\rangle \\ &= |\alpha|^2 \langle Qx, x\rangle + \langle Qy, y\rangle + \alpha\langle Qx, y\rangle + \bar{\alpha}\langle Qy, x\rangle. \end{aligned}$$

The first two terms on the right are zero by assumption. $\alpha = 1$ gives

$$\langle Qx, y\rangle + \langle Qy, x\rangle = 0.$$

$\alpha = i$ gives $\bar{\alpha} = -i$ and

$$\langle Qx, y\rangle - \langle Qy, x\rangle = 0.$$

By addition, $\langle Qx, y\rangle = 0$, and $Q = 0$ follows from (a). ∎

In part (b) of this lemma, it is essential that X be complex. Indeed, the conclusion may not hold if X is real. A counterexample is a rotation Q of the plane $\mathbf{R}^2$ through a right angle. Q is linear, and $Qx \perp x$, hence $\langle Qx, x\rangle = 0$ for all $x \in \mathbf{R}^2$, but $Q \neq 0$. (What about such a rotation in the complex plane?)

We can now list and prove some general properties of Hilbert-adjoint operators which one uses quite frequently in applying these operators.

3.9-4 Theorem (Properties of Hilbert-adjoint operators). *Let H_1, H_2 be Hilbert spaces, $S\colon H_1 \longrightarrow H_2$ and $T\colon H_1 \longrightarrow H_2$ bounded linear operators and α any scalar. Then we have*

(a) $\qquad \langle T^* y, x\rangle = \langle y, Tx\rangle \qquad (x \in H_1,\ y \in H_2)$

(b) $\qquad (S+T)^* = S^* + T^*$

(c) $\qquad (\alpha T)^* = \bar{\alpha} T^*$

(6) (d) $\qquad (T^*)^* = T$

(e) $\qquad \|T^* T\| = \|TT^*\| = \|T\|^2$

(f) $\qquad T^* T = 0 \quad \Longleftrightarrow \quad T = 0$

(g) $\qquad (ST)^* = T^* S^* \qquad$ (assuming $H_2 = H_1$).

Proof. (*a*) From (1) we have (6a):

$$\langle T^*y, x\rangle = \overline{\langle x, T^*y\rangle} = \overline{\langle Tx, y\rangle} = \langle y, Tx\rangle.$$

(*b*) By (1), for all x and y,

$$\begin{aligned}\langle x, (S+T)^*y\rangle &= \langle (S+T)x, y\rangle \\ &= \langle Sx, y\rangle + \langle Tx, y\rangle \\ &= \langle x, S^*y\rangle + \langle x, T^*y\rangle \\ &= \langle x, (S^*+T^*)y\rangle.\end{aligned}$$

Hence $(S+T)^*y = (S^*+T^*)y$ for all y by 3.8-2, which is (6b) by definition.

(*c*) Formula (6c) must not be confused with the formula $T^*(\alpha x) = \alpha T^*x$. It is obtained from the following calculation and subsequent application of Lemma 3.9-3(*a*) to $Q = (\alpha T)^* - \bar{\alpha}T^*$.

$$\begin{aligned}\langle (\alpha T)^*y, x\rangle &= \langle y, (\alpha T)x\rangle \\ &= \langle y, \alpha(Tx)\rangle \\ &= \bar{\alpha}\langle y, Tx\rangle \\ &= \bar{\alpha}\langle T^*y, x\rangle \\ &= \langle \bar{\alpha}T^*y, x\rangle.\end{aligned}$$

(*d*) $(T^*)^*$ is written T^{**} and equals T since for all $x \in H_1$ and $y \in H_2$ we have from (6a) and (1)

$$\langle (T^*)^*x, y\rangle = \langle x, T^*y\rangle = \langle Tx, y\rangle$$

and (6d) follows from Lemma 3.9-3(a) with $Q = (T^*)^* - T$.

(*e*) We see that $T^*T\colon H_1 \longrightarrow H_1$, but $TT^*\colon H_2 \longrightarrow H_2$. By the Schwarz inequality,

$$\|Tx\|^2 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle \leqq \|T^*Tx\|\,\|x\| \leqq \|T^*T\|\,\|x\|^2.$$

Taking the supremum over all x of norm 1, we obtain $\|T\|^2 \leqq \|T^*T\|$. Applying (7), Sec. 2.7, and (2), we thus have

$$\|T\|^2 \leqq \|T^*T\| \leqq \|T^*\| \, \|T\| = \|T\|^2.$$

Hence $\|T^*T\| = \|T\|^2$. Replacing T by T^* and using again (2), we also have

$$\|T^{**}T^*\| = \|T^*\|^2 = \|T\|^2.$$

Here $T^{**} = T$ by (6d), so that (6e) is proved.

(*f*) From (6e) we immediately obtain (6f).

(*g*) Repeated application of (1) gives

$$\langle x, (ST)^*y\rangle = \langle (ST)x, y\rangle = \langle Tx, S^*y\rangle = \langle x, T^*S^*y\rangle.$$

Hence $(ST)^*y = T^*S^*y$ by 3.8-2, which is (6g) by definition. ∎

Problems

1. Show that $0^* = 0$, $I^* = I$.

2. Let H be a Hilbert space and $T\colon H \longrightarrow H$ a bijective bounded linear operator whose inverse is bounded. Show that $(T^*)^{-1}$ exists and

$$(T^*)^{-1} = (T^{-1})^*.$$

3. If (T_n) is a sequence of bounded linear operators on a Hilbert space and $T_n \longrightarrow T$, show that $T_n{}^* \longrightarrow T^*$.

4. Let H_1 and H_2 be Hilbert spaces and $T\colon H_1 \longrightarrow H_2$ a bounded linear operator. If $M_1 \subset H_1$ and $M_2 \subset H_2$ are such that $T(M_1) \subset M_2$, show that $M_1{}^\perp \supset T^*(M_2{}^\perp)$.

5. Let M_1 and M_2 in Prob. 4 be closed subspaces. Show that then $T(M_1) \subset M_2$ if and only if $M_1{}^\perp \supset T^*(M_2{}^\perp)$.

6. If $M_1 = \mathcal{N}(T) = \{x \mid Tx = 0\}$ in Prob. 4, show that

(*a*) $T^*(H_2) \subset M_1{}^\perp$, (*b*) $[T(H_1)]^\perp \subset \mathcal{N}(T^*)$, (*c*) $M_1 = [T^*(H_2)]^\perp$.

7. Let T_1 and T_2 be bounded linear operators on a complex Hilbert space H into itself. If $\langle T_1x, x\rangle = \langle T_2x, x\rangle$ for all $x \in H$, show that $T_1 = T_2$.

8. Let $S = I + T^*T\colon\ H \longrightarrow H$, where T is linear and bounded. Show that $S^{-1}\colon\ S(H) \longrightarrow H$ exists.

9. Show that a bounded linear operator $T\colon\ H \longrightarrow H$ on a Hilbert space H has a finite dimensional range if and only if T can be represented in the form

$$Tx = \sum_{j=1}^{n} \langle x, v_j\rangle w_j \qquad [v_j, w_j \in H].$$

10. (Right shift operator) Let (e_n) be a total orthonormal sequence in a separable Hilbert space H and define the *right shift operator* to be the linear operator $T\colon\ H \longrightarrow H$ such that $Te_n = e_{n+1}$ for $n = 1, 2, \cdots$. Explain the name. Find the range, null space, norm and Hilbert-adjoint operator of T.

3.10 Self-Adjoint, Unitary and Normal Operators

Classes of bounded linear operators of great practical importance can be defined by the use of the Hilbert-adjoint operator as follows.

3.10-1 Definition (Self-adjoint, unitary and normal operators). A bounded linear operator $T\colon\ H \longrightarrow H$ on a Hilbert space H is said to be

self-adjoint or *Hermitian* if	$T^* = T$,
unitary if T is bijective and	$T^* = T^{-1}$,
normal if	$TT^* = T^*T$. ∎

The Hilbert-adjoint operator T^* of T is defined by (1), Sec. 3.9, that is,

$$\langle Tx, y\rangle = \langle x, T^*y\rangle.$$

If T is self-adjoint, we see that the formula becomes

$$\langle Tx, y\rangle = \langle x, Ty\rangle. \tag{1}$$

If T is self-adjoint or unitary, T is normal.

This can immediately be seen from the definition. Of course, a normal operator need not be self-adjoint or unitary. For example, if $I: H \longrightarrow H$ is the identity operator, then $T = 2iI$ is normal since $T^* = -2iI$ (cf. 3.9-4), so that $TT^* = T^*T = 4I$ but $T^* \neq T$ as well as $T^* \neq T^{-1} = -\frac{1}{2}iI$.

Operators which are not normal will easily result from the next example. Another operator which is not normal is T in Prob. 10, Sec. 3.9, as the reader may prove.

The terms in Def. 3.10-1 are also used in connection with matrices. We want to explain the reason for this and mention some important relations, as follows.

3.10-2 Example (Matrices). We consider $\mathbf{C}^n$ with the inner product defined by (cf. 3.1-4)

$$\langle x, y \rangle = x^{\mathsf{T}} \bar{y}, \tag{2}$$

where x and y are written as column vectors, and T means transposition; thus $x^{\mathsf{T}} = (\xi_1, \cdots, \xi_n)$, and we use the ordinary matrix multiplication.

Let $T: \mathbf{C}^n \longrightarrow \mathbf{C}^n$ be a linear operator (which is bounded by Theorem 2.7-8). A basis for $\mathbf{C}^n$ being given, we can represent T and its Hilbert-adjoint operator T^* by two n-rowed square matrices, say, A and B, respectively.

Using (2) and the familiar rule $(Bx)^{\mathsf{T}} = x^{\mathsf{T}} B^{\mathsf{T}}$ for the transposition of a product, we obtain

$$\langle Tx, y \rangle = (Ax)^{\mathsf{T}} \bar{y} = x^{\mathsf{T}} A^{\mathsf{T}} \bar{y}$$

and

$$\langle x, T^* y \rangle = x^{\mathsf{T}} \bar{B} \bar{y}.$$

By (1), Sec. 3.9, the left-hand sides are equal for all $x, y \in \mathbf{C}^n$. Hence we must have $A^{\mathsf{T}} = \bar{B}$. Consequently,

$$B = \bar{A}^{\mathsf{T}}.$$

Our result is as follows.

If a basis for $\mathbf{C}^n$ is given and a linear operator on $\mathbf{C}^n$ is represented by a certain matrix, then its Hilbert-adjoint operator is represented by the complex conjugate transpose of that matrix.

Consequently, representing matrices are

Hermitian if T is self-adjoint (Hermitian),
unitary if T is unitary,
normal if T is normal.

Similarly, for a linear operator $T: \mathbf{R}^n \longrightarrow \mathbf{R}^n$, representing matrices are:

Real symmetric if T is self-adjoint,
orthogonal if T is unitary.

In this connection, remember the following definitions. A square matrix $A = (\alpha_{jk})$ is said to be:

Hermitian if $\bar{A}^{\mathsf{T}} = A$ (hence $\bar{\alpha}_{kj} = \alpha_{jk}$)
skew-Hermitian if $\bar{A}^{\mathsf{T}} = -A$ (hence $\bar{\alpha}_{kj} = -\alpha_{jk}$)
unitary if $\bar{A}^{\mathsf{T}} = A^{-1}$
normal if $A\bar{A}^{\mathsf{T}} = \bar{A}^{\mathsf{T}}A$.

A *real* square matrix $A = (\alpha_{jk})$ is said to be:

(*Real*) *symmetric* if $A^{\mathsf{T}} = A$ (hence $\alpha_{kj} = \alpha_{jk}$)
(*real*) *skew-symmetric* if $A^{\mathsf{T}} = -A$ (hence $\alpha_{kj} = -\alpha_{jk}$)
orthogonal if $A^{\mathsf{T}} = A^{-1}$.

Hence a real Hermitian matrix is a (real) symmetric matrix. A real skew-Hermitian matrix is a (real) skew-symmetric matrix. A real unitary matrix is an orthogonal matrix. (Hermitian matrices are named after the French mathematician, Charles Hermite, 1822–1901.) ∎

Let us return to linear operators on arbitrary Hilbert spaces and state an important and rather simple criterion for self-adjointness.

3.10-3 Theorem (Self-adjointness). *Let $T: H \longrightarrow H$ be a bounded linear operator on a Hilbert space H. Then:*

(a) *If T is self-adjoint, $\langle Tx, x\rangle$ is real for all $x \in H$.*

(b) *If H is complex and $\langle Tx, x\rangle$ is real for all $x \in H$, the operator T is self-adjoint.*

Proof. **(a)** If T is self-adjoint, then for all x,

$$\overline{\langle Tx, x\rangle} = \langle x, Tx\rangle = \langle Tx, x\rangle.$$

Hence $\langle Tx, x\rangle$ is equal to its complex conjugate, so that it is real.

(b) If $\langle Tx, x\rangle$ is real for all x, then

$$\langle Tx, x\rangle = \overline{\langle Tx, x\rangle} = \overline{\langle x, T^*x\rangle} = \langle T^*x, x\rangle.$$

Hence

$$0 = \langle Tx, x\rangle - \langle T^*x, x\rangle = \langle (T - T^*)x, x\rangle$$

and $T - T^* = 0$ by Lemma 3.9-3(b) since H is complex. ∎

In part (b) of the theorem it is essential that H be complex. This is clear since for a real H the inner product is real-valued, which makes $\langle Tx, x\rangle$ real without any further assumptions about the linear operator T.

Products (composites[6]) of self-adjoint operators appear quite often in applications, so that the following theorem will be useful.

3.10-4 Theorem (Self-adjointness of product). *The product of two bounded self-adjoint linear operators S and T on a Hilbert space H is self-adjoint if and only if the operators commute,*

$$ST = TS.$$

Proof. By (6g) in the last section and by the assumption,

$$(ST)^* = T^*S^* = TS.$$

Hence

$$ST = (ST)^* \qquad \Longleftrightarrow \qquad ST = TS.$$

This completes the proof. ∎

Sequences of self-adjoint operators occur in various problems, and for them we have

[6] A review of terms and notations in connection with the composition of mappings is included in A1.2, Appendix 1.