3.10-5 Theorem (Sequences of self-adjoint operators). *Let (T_n) be a sequence of bounded self-adjoint linear operators $T_n: H \longrightarrow H$ on a Hilbert space H. Suppose that (T_n) converges, say,*

$$T_n \longrightarrow T, \qquad \text{that is,} \qquad \|T_n - T\| \longrightarrow 0,$$

where $\|\cdot\|$ is the norm on the space $B(H, H)$; cf. Sec. 2.10. Then the limit operator T is a bounded self-adjoint linear operator on H.

Proof. We must show that $T^* = T$. This follows from $\|T - T^*\| = 0$. To prove the latter, we use that, by 3.9-4 and 3.9-2,

$$\|T_n^* - T^*\| = \|(T_n - T)^*\| = \|T_n - T\|$$

and obtain by the triangle inequality in $B(H, H)$

$$\begin{aligned} \|T - T^*\| &\leqq \|T - T_n\| + \|T_n - T_n^*\| + \|T_n^* - T^*\| \\ &= \|T - T_n\| + 0 + \|T_n - T\| \\ &= 2\,\|T_n - T\| \qquad \longrightarrow \qquad 0 \qquad\qquad (n \longrightarrow \infty). \end{aligned}$$

Hence $\|T - T^*\| = 0$ and $T^* = T$. ∎

These theorems give us some idea about basic properties of self-adjoint linear operators. They will also be helpful in our further work, in particular in the spectral theory of these operators (Chap. 9), where further properties will be discussed.

We now turn to unitary operators and consider some of their basic properties.

3.10-6 Theorem (Unitary operator). *Let the operators $U: H \longrightarrow H$ and $V: H \longrightarrow H$ be unitary; here, H is a Hilbert space. Then:*

(a) *U is isometric* (cf. 1.6-1); *thus $\|Ux\| = \|x\|$ for all $x \in H$;*

(b) $\|U\| = 1$, *provided $H \neq \{0\}$,*

(c) $U^{-1} (= U^*)$ *is unitary,*

(d) *UV is unitary,*

(e) *U is normal.*

Furthermore:

(f) *A bounded linear operator T on a complex Hilbert space H is unitary if and only if T is isometric and surjective.*

Proof. **(a)** can be seen from

$$\|Ux\|^2 = \langle Ux, Ux\rangle = \langle x, U^*Ux\rangle = \langle x, Ix\rangle = \|x\|^2.$$

(b) follows immediately from (a).

(c) Since U is bijective, so is U^{-1}, and by 3.9-4,

$$(U^{-1})^* = U^{**} = U = (U^{-1})^{-1}.$$

(d) UV is bijective, and 3.9-4 and 2.6-11 yield

$$(UV)^* = V^*U^* = V^{-1}U^{-1} = (UV)^{-1}.$$

(e) follows from $U^{-1} = U^*$ and $UU^{-1} = U^{-1}U = I$.

(f) Suppose that T is isometric and surjective. Isometry implies injectivity, so that T is bijective. We show that $T^* = T^{-1}$. By the isometry,

$$\langle T^*Tx, x\rangle = \langle Tx, Tx\rangle = \langle x, x\rangle = \langle Ix, x\rangle.$$

Hence

$$\langle (T^*T - I)x, x\rangle = 0$$

and $T^*T - I = 0$ by Lemma 3.9-3(b), so that $T^*T = I$. From this,

$$TT^* = TT^*(TT^{-1}) = T(T^*T)T^{-1} = TIT^{-1} = I.$$

Together, $T^*T = TT^* = I$. Hence $T^* = T^{-1}$, so that T is unitary. The converse is clear since T is isometric by (a) and surjective by definition. ∎

Note that an isometric operator need not be unitary since it may fail to be surjective. An example is the *right shift operator* $T\colon\ l^2 \longrightarrow l^2$ given by

$$(\xi_1, \xi_2, \xi_3, \cdots) \quad \longmapsto \quad (0, \xi_1, \xi_2, \xi_3, \cdots)$$

where $x = (\xi_j) \in l^2$.

Problems

1. If S and T are bounded self-adjoint linear operators on a Hilbert space H and α and β are real, show that $\tilde{T} = \alpha S + \beta T$ is self-adjoint.

2. How could we use Theorem 3.10-3 to prove Theorem 3.10-5 for a complex Hilbert space H?

3. Show that if $T\colon H \longrightarrow H$ is a bounded self-adjoint linear operator, so is T^n, where n is a positive integer.

4. Show that for any bounded linear operator T on H, the operators

$$T_1 = \frac{1}{2}(T + T^*) \qquad \text{and} \qquad T_2 = \frac{1}{2i}(T - T^*)$$

are self-adjoint. Show that

$$T = T_1 + iT_2, \qquad\qquad T^* = T_1 - iT_2.$$

Show uniqueness, that is, $T_1 + iT_2 = S_1 + iS_2$ implies $S_1 = T_1$ and $S_2 = T_2$; here, S_1 and S_2 are self-adjoint by assumption.

5. On $\mathbf{C}^2$ (cf. 3.1-4) let the operator $T\colon \mathbf{C}^2 \longrightarrow \mathbf{C}^2$ be defined by $Tx = (\xi_1 + i\xi_2, \xi_1 - i\xi_2)$, where $x = (\xi_1, \xi_2)$. Find T^*. Show that we have $T^*T = TT^* = 2I$. Find T_1 and T_2 as defined in Prob. 4.

6. If $T\colon H \longrightarrow H$ is a bounded self-adjoint linear operator and $T \neq 0$, then $T^n \neq 0$. Prove this (*a*) for $n = 2, 4, 8, 16, \cdots$, (*b*) for every $n \in \mathbf{N}$.

7. Show that the column vectors of a unitary matrix constitute an orthonormal set with respect to the inner product on $\mathbf{C}^n$.

8. Show that an isometric linear operator $T\colon H \longrightarrow H$ satisfies $T^*T = I$, where I is the identity operator on H.

9. Show that an isometric linear operator $T\colon H \longrightarrow H$ which is not unitary maps the Hilbert space H onto a proper closed subspace of H.

10. Let X be an inner product space and $T\colon X \longrightarrow X$ an isometric linear operator. If $\dim X < \infty$, show that T is unitary.

11. (Unitary equivalence) Let S and T be linear operators on a Hilbert space H. The operator S is said to be *unitarily equivalent* to T if there is a unitary operator U on H such that

$$S = UTU^{-1} = UTU^*.$$

If T is self-adjoint, show that S is self-adjoint.

12. Show that T is normal if and only if T_1 and T_2 in Prob. 4 commute. Illustrate part of the situation by two-rowed normal matrices.

13. If $T_n\colon H \longrightarrow H$ $(n = 1, 2, \cdots)$ are normal linear operators and $T_n \longrightarrow T$, show that T is a normal linear operator.

14. If S and T are normal linear operators satisfying $ST^* = T^*S$ and $TS^* = S^*T$, show that their sum $S + T$ and product ST are normal.

15. Show that a bounded linear operator $T\colon H \longrightarrow H$ on a complex Hilbert space H is normal if and only if $\|T^*x\| = \|Tx\|$ for all $x \in H$. Using this, show that for a normal linear operator,

$$\|T^2\| = \|T\|^2.$$