3.

$$\lim_{(x,y)\longrightarrow(2,-1)} \left[\frac{x+y}{x-y}\right]^3$$

4.

$$\lim_{(x,y)\longrightarrow(0,0)} \frac{x^2-xy}{\sqrt{x}-\sqrt{y}}$$

Solution

1.

$$\lim_{(x,y)\longrightarrow(\sqrt{2},0)} x^2 + \lim_{(x,y)\longrightarrow(\sqrt{2},0)} xy = (\sqrt{2})^2 + (\sqrt{2})(0) = 2+0 = 2$$

2.

$$\lim_{(x,y)\longrightarrow(0,0)} \sin(xy) - \lim_{(x,y)\longrightarrow(0,0)} \cos(y^2) = \sin(0) - \cos(0) = -1$$

3.

$$\left[\frac{\lim_{(x,y)\longrightarrow(2,-1)} x+y}{\lim_{(x,y)\longrightarrow(2,-1)} x-y}\right]^3 = \left(\frac{1}{3}\right)^3 = \frac{1}{27}$$

4.

$$\lim_{(x,y)\longrightarrow(0,0)} \frac{x(x-y)}{\sqrt{x}-\sqrt{y}} = \lim_{(x,y)\longrightarrow(0,0)} \frac{x(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})}{\sqrt{x}-\sqrt{y}} = 0(\sqrt{0}+\sqrt{0}) = 0$$

1.3 Partial Differentiation

Consider the function $z = f(x,y)$, and let x change to $x+\Delta x$, while y remains constant. In this case, z will change to $z+\Delta z$, so that

$$\begin{aligned} z+\Delta z &= f(x+\Delta x, y), \\ \Delta z &= f(x+\Delta x, y) - z, \\ \Delta z &= f(x+\Delta x, y) - f(x,y), \\ \frac{\Delta z}{\Delta x} &= \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}. \end{aligned}$$

Now, upon taking the limit as Δx goes to zero, we have the partial derivative of z with respect to x:

$$\frac{\partial z}{\partial x} = \lim_{\Delta x\longrightarrow 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}.$$

Note that the partial derivative can be denoted by either

$$\frac{\partial z}{\partial x} \text{ or } \frac{\partial f}{\partial x} \text{ or } Z_x \text{ or } f_x(x, y).$$

Similarly, we have for $\frac{\partial z}{\partial y}$

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \longrightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}.$$

Note that the partial derivative can be denoted by either

$$\frac{\partial z}{\partial y} \text{ or } \frac{\partial f}{\partial y} \text{ or } Z_y \text{ or } f_y(x, y).$$

Example: Using the limit definition of partial derivative to find $\frac{\partial z}{\partial x}$ of the function $f(x, y) = xy$? Solution

$$\begin{aligned}
\frac{\partial z}{\partial x} &= \lim_{\Delta x \longrightarrow 0} \frac{(x + \Delta x)y - xy}{\Delta x}, \\
&= \lim_{\Delta x \longrightarrow 0} \frac{xy + y\Delta x - xy}{\Delta x}, \\
&= \lim_{\Delta x \longrightarrow 0} \frac{y\Delta x}{\Delta x}, \\
&= y.
\end{aligned}$$

Example: Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$?

a- $f(x, y) = x^2 + 3xy + y - 1$

b- $f(x, y) = y \sin(xy)$

c- $f(x, y) = \sqrt{x^2 + y^2}$

Solution

a- $\frac{\partial z}{\partial x} = 2x + 3y$ and $\frac{\partial z}{\partial y} = 3x + 1$

b- $\frac{\partial z}{\partial x} = y^2 \cos(xy)$ and $\frac{\partial z}{\partial y} = xy \cos(xy) + \sin(xy)$

c- $\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}$ and $\frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}$

Example: Given $f(x, y) = 3x^2 - 2xy^2 + 2$, find $f_x(-3, 0)$ and $f_y(2, -1)$? Solution

We have $f_x = 6x - 2y^2$ and $f_y = -4xy$ so $f_x(-3, 0) = 6(-3) - 2(0)^2 = -18$ and $f_y(2, -1) = -4(2)(-1) = 8$.