1.3.1 Higher Order Derivatives

We can form the second order derivative with respect to x where
$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$.
Similarly, we can form second order derivative with respect to y where
$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$ and $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}$.
Note that in general, $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$.
$\frac{\partial^2 z}{\partial x^2} = z_{xx}$ and $\frac{\partial^2 z}{\partial x \partial y} = z_{xy}$
Similarly, $\frac{\partial^2 z}{\partial y^2} = z_{yy}$ and $\frac{\partial^2 z}{\partial y \partial x} = z_{yx}$
Example: Verify that $w_{yx} = w_{xy}$ if $w(x, y) = x^2 - xy + y^2$?
Solution
We have $w_x = 2x - y$ and $w_{xy} = -1$. We have also $w_y = -x + 2y$ and $w_{yx} = -1$. Thus, $w_{yx} = -1 = w_{xy}$
Example: Find $\frac{\partial^4 f}{\partial s \partial r \partial s \partial t}$ if $f(r, s, t) = 1 - 2rs^2t + r^2s$?
Solution

We first differentiate with respect to the variable s, then r, then s again , and finally with respect to t. We have

$$\begin{aligned} \frac{\partial f}{\partial s} &= -4rst + r^2, \\ \frac{\partial^2 f}{\partial s \partial r} &= -4st + 2r, \\ \frac{\partial^3 f}{\partial s \partial r \partial s} &= -4t, \\ \frac{\partial^4 f}{\partial s \partial r \partial s \partial t} &= -4. \end{aligned}$$

Example: If $z = e^{x^2+y^2}$, then show that $yz_x - xz_y = 0$?
Solution
We first need to find z_x and z_y. So, $z_x = 2xe^{x^2+y^2}$ and $z_y = 2ye^{x^2+y^2}$. Thus, by substituting into $yz_x - xz_y$ we get

$$\begin{aligned} yz_x - xz_y &= 2yxe^{x^2+y^2} - 2yxe^{x^2+y^2}, \\ &= 0. \end{aligned}$$

Example: If $z = f(x + cy) + g(x - cy)$, then show that $c^2 z_{xx} - z_{yy} = 0$?
Solution

Assume $u = x + cy$ and $v = x - cy$ so we have $u_x = 1, v_x = 1$ and $u_y = c, v_y = -c$. To find z_{xx}

$$\begin{aligned} z_x &= f'(u)u_x + g'(v)v_x \\ &= f'(x+cy) + g'(x-cy). \end{aligned}$$

Also,

$$\begin{aligned} z_{xx} &= f''(u)u_x + g''(v)v_x \\ &= f''(x+cy) + g''(x-cy). \end{aligned}$$

Now, to find z_{yy}

$$\begin{aligned} z_y &= f'(u)u_y + g'(v)v_y \\ &= cf'(x+cy) - cg'(x-cy). \end{aligned}$$

Also,

$$\begin{aligned} z_{yy} &= cf''(u)u_x - cg''(v)v_x \\ &= c^2f''(x+cy) + c^2g''(x-cy). \end{aligned}$$

Thus, by substituting into $c^2z_{xx} - z_{yy}$ we get

$$\begin{aligned} c^2z_{xx} - z_{yy} &= c^2f''(x+cy) + c^2g''(x-cy) - c^2f''(x+cy) - c^2g''(x-cy). \\ &= 0. \end{aligned}$$

Example: Consider a function $T = \ln(\sqrt{r^2+s^2})$. Prove that $r\frac{\partial T}{\partial r} + s\frac{\partial T}{\partial s} = 1$.
Solution

We know that $T = \frac{1}{2}\ln(r^2+s^2)$ then

$$\begin{aligned} \frac{\partial T}{\partial r} &= \frac{1}{2}\left[\frac{2r}{r^2+s^2}\right] \\ &= \frac{r}{r^2+s^2}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial T}{\partial s} &= \frac{1}{2}\left[\frac{2s}{r^2+s^2}\right] \\ &= \frac{s}{r^2+s^2}. \end{aligned}$$

Thus, by substituting into $r\frac{\partial T}{\partial r} + s\frac{\partial T}{\partial s}$

$$r\left[\frac{r}{r^2+s^2}\right] + s\left[\frac{s}{r^2+s^2}\right] = \frac{r^2+s^2}{r^2+s^2} = 1.$$